Linear Matrix Inequalities Based State Feedback and Reference Feedforward Actuator Saturated H-infinity Control of Small-Scale Unmanned Helicopter

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Abstract

In this study, systematic procedure of the flight controller design for a small-scale unmanned helicopter is presented. The procedure is based on a linear dynamical model. The proposed controller is composed of state feedback and reference feedforward. Reference tracking performance is formulated in terms of L_2 gain from reference inputs to tracking errors and respective integral terms. Solution of the optimal controller with minimum L_2 gain is cast to the semi definite programming problem with a set of Linear Matrix Inequality (LMI) constraints. Six degree-of-freedom linear helicopter model with two degree of freedom rotor dynamics is used to illustrate the effectiveness of approach through simulations. Numerical simulations show that the stability of controlled system and boundedness of control signals against reference trajectories with bounded magnitudes are guaranteed by the proposed controller.

1. Introduction

Unmanned Aerial Vehicles (UAV) have seen unprecedented levels of development over the last decade. It is well known that UAVs will be used in the future comprehensively for civilian and military applications such as environmental monitoring, power line inspection, surveillance, search and rescue etc. From all classes of UAVs, unmanned rotorcrafts, and in particular unmanned helicopters, have superiorities over fixed wing UAVs because they take-off and land vertically, they do not require a runway, and they are able to hover and fly in very low altitudes [1].

The flight controller is essential for a UAV to achieve autonomous flight missions [2]. A large variety of attempts that have been reported in literature to develop flight controllers using various algorithms. Optimal linear quadratic controllers are designed in previous researchs [3-5]. Robust and multi loop PID controllers proposed for autonomous flight [6-7]. Neural network approach offered by several researchers to obtain adaptive controllers [8-10]. Isodori et al. used the differential geometry method to combine adaptive and robust control structures [11]. The robust and H_∞ control techniques applied by various researchers [12-15]. The composite nonlinear feedback control with decoupling approach considered as a potential solution by Peng et al. [16].

In this study, a new H_∞ controller with state feedback and reference feedforward is proposed for reference tracking. To avoid actuator saturation problem, boundedness of control signals against magnitude bounded reference inputs is formulated by LMIs. In order to examine the performance of proposed controller by numerical simulations, parameterized linear model of Raptor 90 SE is used.

Rest of the paper is organized as follows: Section 2 describes the synthesis of the proposed controller. Numerical simulation results are given in Section 3. Finally, Section 4 concludes the paper.

2. LMI Based State Feedback and Reference Feedforward Actuator Saturated H_∞ Controller

In this section, we consider an optimal state feedback and reference feedforward actuator saturated H_∞ controller synthesis problem. Structure of the controller can be seen in Fig. 1.

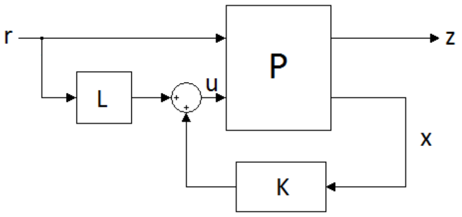

Fig. 1. Controller Structure

Consider a Linear Time Invariant (LTI) system

$$\dot{x} = Ax + Bu \quad (1)$$

where $x \in \Re^n$ is a state vector, $u \in \Re^m$ is a control input vector, $A \in \Re^{nxn}$ is a state matrix and $B \in \Re^{nxm}$ is a control input matrix. To design a tracking controller, state space system (1) should be augmented

$$\begin{aligned} \dot{x}_a &= A_a x_a + B_1 r + B_2 u \\ z &= C x_a + D_1 r + D_2 u \end{aligned} \quad (2)$$

where $r \in \Re^p$ is a vector of reference trajectories and $z \in \Re^c$ is a vector of controlled outputs, $A_a \in \Re^{(n+p)x(n+p)}$ is the augmented state matrix, $B_1 \in \Re^{(n+p)xp}$ is the reference inputs matrix, $B_2 \in \Re^{(n+p)xm}$ is the control inputs matrix and C , D_1, D_2 are the matrices with appropriate dimensions to construct controlled output vector. Assume that, the first p element of given state vector $x_t \in \Re^p$ are the states supposed to track reference trajectories. Hence the state space system (2) can be written as

$$\underbrace{\begin{bmatrix}\dot{x}\\ r-x_t\end{bmatrix}}_{\dot{x}_a} = \underbrace{\begin{bmatrix}A_{nxn} & 0_{nxp}\\ -I_{pxp} & 0_{pxn}\end{bmatrix}}_{A_a}\underbrace{\begin{bmatrix}x\\ \int r-x_t\end{bmatrix}}_{x_a} + \underbrace{\begin{bmatrix}0\\ I\end{bmatrix}}_{B_1} r + \underbrace{\begin{bmatrix}B\\ 0\end{bmatrix}}_{B_2} u \qquad (3)$$

For a control law which is a linear function of x and r

$$u = Kx + Lr \qquad (4)$$

where $K \in \Re^{mxn}$ and $L \in \Re^{mxp}$ are the controller gain matrices with appropriate dimensions. Closed loop system can be written

$$\begin{aligned}\dot{x}_a &= (A_a + B_2K)x_a + (B_1 + B_2L)r\\ z &= (C + D_2K)x_a + (D_1 + D_2L)r\end{aligned} \qquad (5)$$

There exists a positive definite quadratic Lyapunov function $V(x_a) = x_a{}^T P x_a$ where $P = P^T \succ 0$ and its negative definite derivative for any stable LTI system. H_∞ performance problem is to find an controller that makes L_2 gain of the closed loop system from reference inputs to controlled outputs, less than a positive scalar γ. If minimization of γ is achieved, then the computed controller is an optimal H_∞ controller. Stability and L_2 gain properties of a LTI system can be expressed simultaneously by a single inequality which is called Hamiltonian of the system

$$\dot{V}(x_a) + z^T z - \gamma^2 r^T r < 0 \qquad (6)$$

Although the stable controller design with L_2 gain of γ considered, the actuator saturation problem has not been taken into account yet. In order to meet the problem, assume that the reference input vector belongs to the following set

$$W = \{r \in \Re^p; r^T R r \le 1\} \qquad (7)$$

with $R = R^T \succ 0$. In this case, r is bounded by a quadratic norm which reflects bounds on r [17-19]. Note that, if R is an diagonal matrix, it denotes that $|r_i| \le \sqrt{1/R_i}$, where R_i represents the i th diagonal element of R, $i = 1,\ldots,p$ [17]. It is well known that, quadratic Lyapunov functions constructs an invariant ellipsoid for LTI systems. Since quadratic Lyapunov function is positive definite and its derivative is negative definite, the state trajectories that initialized in ellipsoid

$$x_a{}^T P x_a \le 1 \qquad (8)$$

do not escape from this domain [20]. Consider Hamiltonian of the system (6) and the application of S-procedure [21], a sufficient inequality to relate (6), (7) and (8) is obtained.

$$\dot{V}(x_a) + z^T z - \gamma^2 r^T r + \tau_1(x_a{}^T P x_a - 1) + \tau_2(1 - r^T R r) < 0 \qquad (9)$$

with positive scalars τ_1, τ_2. In particular, if (10) and (11) are satisfied,

$$\dot{V}(x_a) + z^T z - \gamma^2 r^T r + \tau_1 x_a{}^T P x_a - \tau_2 r^T R r < 0 \qquad (10)$$

$$\tau_2 < \tau_1 \qquad (11)$$

Thus inequality (9) also holds [17]. Hence, this can be concluded that (9) ensures that the trajectories initialized in ellipsoid (8), stays in this ellipsoid (7). Arranging the inequality (10), the following matrix inequality can be obtained as

$$\begin{bmatrix}x\\ r\end{bmatrix}^T \varphi \begin{bmatrix}x\\ r\end{bmatrix} < 0 \qquad (12)$$

$$\varphi \prec 0 \qquad (13)$$

Since $u = Kx + Lr$, one can always write

$$\|u\|_2 \le u_{max} \leftrightarrow u^T u \le u_{max}^2 \qquad (14)$$

For simplicity, let assume that $u_{max} = 1$, by scaling B_2. Hence (12) can be rewritten as an ellipsoid

$$(Kx + Lr)^T (Kx + Lr) \le 1 \qquad (15)$$

$$\begin{bmatrix}x\\ r\end{bmatrix}^T \begin{bmatrix}K^T K & K^T L\\ L^T K & L^T L\end{bmatrix}\begin{bmatrix}x\\ r\end{bmatrix} \le 1 \qquad (16)$$

which is a set of the state trajectories and reference inputs do not cause actuator saturation. To avoid actuator saturation ellipsoid (14) must contain the union of ellipsoids (7) and (8) which is written below

$$\begin{bmatrix}x\\ r\end{bmatrix}^T \begin{bmatrix}P & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}x\\ r\end{bmatrix} \le 1 \qquad (17)$$

$$\begin{bmatrix}K^T K & K^T L\\ L^T K & L^T L\end{bmatrix} < \begin{bmatrix}P & 0\\ 0 & R\end{bmatrix} \qquad (18)$$

$$\begin{bmatrix}P - K^T K & -K^T L\\ -L^T K & R - L^T L\end{bmatrix} > 0 \qquad (19)$$

Problem of designing a state feedback and reference feedforward actuator saturated H_∞ controller can be formulated by Bilinear Matrix Inequalities (BMI) (13) and (19). By applying congruence transformation [21] pre and post multiply these BMIs by

$$\begin{bmatrix}X & 0\\ 0 & I\end{bmatrix} \qquad (20)$$

where $X = X^T = P^{-1}$. Then, BMIs (13) and (19) are converted to

$$\overline{\varphi} \prec 0 \qquad (21)$$

$$\begin{bmatrix} X - W^T W & -W^T L \\ -L^T W & R - L^T L \end{bmatrix} \succ 0 \qquad (22)$$

with $W := KX$. The resulting matrix inequalities are still in the form of BMI. Therefore, by applying Schur complement formula [21], expression (21) and (22) are equivalent to

$$\begin{bmatrix} \underbrace{XA_a^T + W^T B_2^T}_{\Omega} + \Omega^T + \tau_1 X & * & * \\ B_1^T + L^T B_2^T & -\gamma^2 I - \tau_2 R & * \\ CX + D_2 W & D_1 + D_2 L & -I \end{bmatrix} \prec 0 \qquad (23)$$

$$\begin{bmatrix} X & 0 & W^T \\ 0 & R & L^T \\ W & L & I \end{bmatrix} \succ 0 \qquad (24)$$

Finally, controller design is formulated with LMIs (23) and (24). The following theorem summarizes the state feedback and reference feedforward actuator saturated H_∞ controller design as a convex optimization problem.

Theorem:

The control law $u = Kx + Lr$ where $K = WX^{-1}$ and L are optimal state feedback and reference feedforward H_∞ controller gains for closed loop system

$$\begin{aligned} \dot{x}_a &= A_a x_a + B_1 r + B_2 u \\ z &= C x_a + D_1 r + D_2 u \end{aligned}$$

with an actuator saturation constraint

$$\|u\|_2 \le u_{max} \leftrightarrow u^T u \le u_{max}^2$$

If and only if there exists symmetric and positive definite matrix $X = X^T \in \Re^{n+p}$ and a rectangular matrix $W \in \Re^{mx(n+p)}$ which satisfy the following optimization problem, for a given τ_1 and R

minimize γ subject to

$$\begin{bmatrix} \underbrace{XA_a^T + W^T B_2^T}_{\Omega} + \Omega^T + \tau_1 X & * & * \\ B_1^T + L^T B_2^T & -\gamma^2 I - \tau_2 R & * \\ CX + D_2 W & D_1 + D_2 L & -I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} X & 0 & W^T \\ 0 & R & L^T \\ W & L & I \end{bmatrix} \succ 0$$

$$\tau_2 < \tau_1$$

3. Numerical Simulations

In this section, simulations are carried out in order to illustrate the effectiveness of proposed controller in reference tracking. All the simulations and computations are accomplished using MATLAB with SIMULINK. For the solution of resulting LMIs, YALMIP Parser and SEDUMI solver are used [22-23]. When τ_1 is fixed to 14 and R is given as (25), γ is computed as 0.0218.

$$R = \mathrm{diag}\left(1/10^2 \quad 1/4^2 \quad 1/4^2 \quad 1/(\pi/4)^2\right) \qquad (25)$$

For simulation studies and controller design, parameterized linear model of the Raptor 90 SE is considered. This model structure is a very appropriate for controller design and simulation since the ability of establish a generic solution to the small-scale helicopter identification problem is approved by literature [1]. The linear parameterized model is based on Mettler' s model for the Carneige Mellon' s Yamaha R-50 and MIT' s X-Cell 60 [24]. The structure of the model proposed by Mettler has been already successfully used for the parametric identification of several helicopters of different sizes and specifications [4-5], [12], [15], [25-26]. The related state space is given as follows

$$\dot{x} = Ax + Bu \qquad (26)$$

where

$$x = \begin{bmatrix} u & v & \theta & \phi & q & p & a & b & w & r & \psi \end{bmatrix}^T \qquad (27)$$

$$u = \begin{bmatrix} u_{lon} & u_{lat} & u_{col} & \overline{u}_{lat} \end{bmatrix}^T \qquad (28)$$

$$A = \begin{bmatrix} X_u & 0 & -g & 0 & 0 & 0 & X_a & 0 & 0 & 0 & 0 \\ 0 & Y_v & 0 & g & 0 & 0 & 0 & Y_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ M_u & M_v & 0 & 0 & 0 & 0 & M_a & 0 & 0 & 0 & 0 \\ L_u & L_v & 0 & 0 & 0 & 0 & 0 & L_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1/\tau_f & A_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & B_a & -1/\tau_f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Z_a & Z_b & Z_w & Z_r & 0 \\ 0 & N_v & 0 & 0 & 0 & N_p & 0 & 0 & N_w & N_r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad (29)$$

$$
B=\begin{bmatrix}
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
A_{lon} & A_{lat} & 0 & 0\\
B_{lon} & B_{lat} & 0 & 0\\
0 & 0 & Z_{col} & 0\\
0 & 0 & N_{col} & N_{ped}\\
0 & 0 & 0 & 0
\end{bmatrix} \tag{30}
$$

where u, v, w are the linear velocities respect to Body Fixed Frame (BFF). p, q, r are the angular velocities respect to Body Fixed Frame (BFF). ϕ, θ, ψ are the euler angles. a and b are states of the first order rotor flapping dynamics and u_{lon}, u_{lat}, u_{col}, $\bar{u}_{lat}$ are the control inputs which are scaled to ±1 [1].

The values of state space model parameters can be found in [1]. Proposed controller is tested for trajectory tracking of Body Fixed Frame (BFF) linear velocities and yaw angle. For trajectory generation from reference inputs in the form of step, linear second order critically damped reference models with $2\ rad/s$ bandwidth are used as shown in Fig. 2.

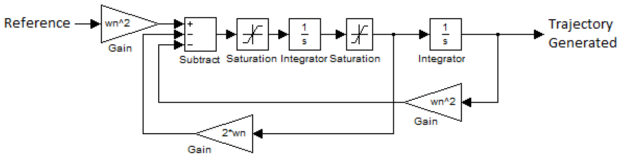

Fig. 2. Second order reference models for trajectory generation

Fig. 3 shows the trajectories generated by second order reference models. The trajectory generation procedure enables smooth tracking performance.

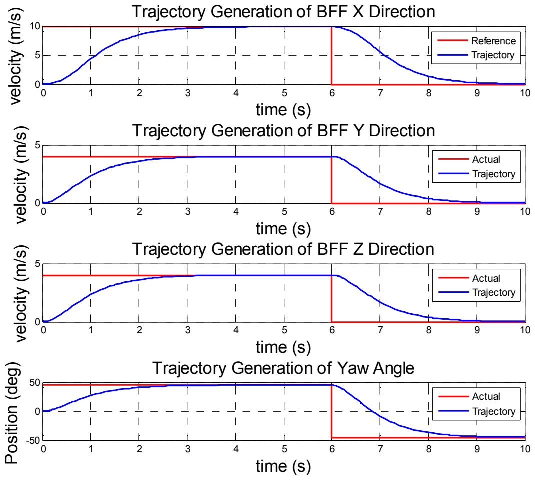

Fig. 3. Trajectory generation from step reference imputs

BFF velocity and yaw angle tracking control performances are shown in Fig. 4. Generated trajectories are tracked successfully without any overshoot.

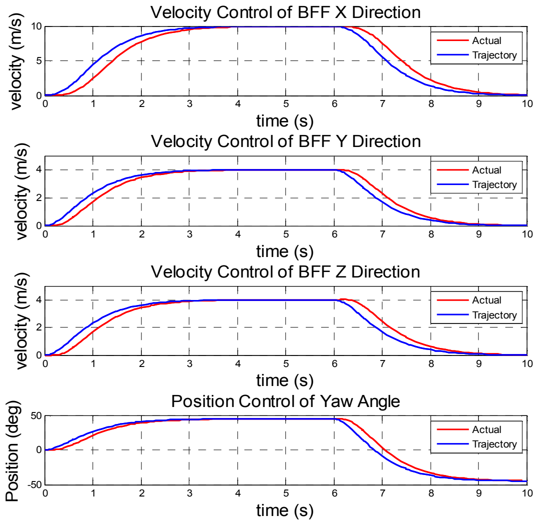

Fig. 4. Reference tracking performance

Time histories of control inputs during tracking can be seen in Fig. 4. Actuator saturation problem is not occurred since the signal magnitudes do not exceed ±1 .

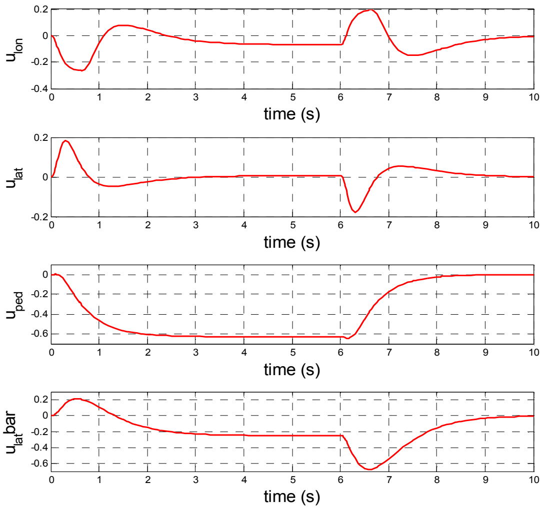

Fig. 5. Time history of control inputs

Resulted pitch and roll angles along trajectory tracking are shown in Fig. 6.

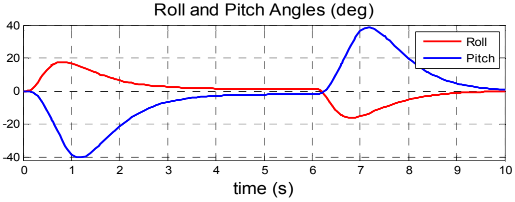

Fig. 6. Roll and pitch angle time histories during reference tracking

4. Conclusions

In this study, systematic procedure of the flight controller design for small-scale unmanned helicopters is presented. The proposed flight controller is composed of state feedback and reference feedforward. The controller design problem is formulated as a convex optimization problem with LMIs. Parameterized linear state space model is used for both simulation and design studies. Numerical simulation results demonstrate that the proposed controller can track reference trajectories without any overshoot and more importantly the magnitudes of control signals stay in adequate level to avoid saturation. Consequently, the proposed controller has a great potential in systematic tracking controller design for small-scale unmanned helicopters which can be represented by linear parameterized dynamic models.

5. References

[1] I. A. Raptis and K. P. Valavanis, "Linear and nonlinear control of small-scale unmanned helicopters", Springer, Dodrecht, Heidelberg, London, New york, 2011.

[2] G. Cai, B. M. Chen and T. H. Lee, "Unmanned rotorcrafts ", Springer, Dodrecht, Heidelberg, London, New york, 2011.

[3] K. Nonami, F. Kendoul, S. Suzuki, W. Wang and D. Nakazawa, "Autonomous flying robots: unmanned aerial vehicles and micro aerial vehicles ", Springer, Tokio, Dodrecht, Heidelberg, London, New york, 2010.

[4] A. Budiyonoa and S. S. Wibowob, "Optimal tracking controller design for a small-scale helicopter", *J Bionic Eng*, vol. *4,* no. *4,* pp. *271-280,* Dec., 2007.

[5] J. Shin, K. Nonami, D. Fujiwara and K. Hazawa, "Model based optimal attitude and positioning control of small-scale unmanned helicopter", *Robotica*, vol. *23,* no. *1,* pp. *51-63,* Jan., 2005.

[6] T. Sakamato, H. Katamaya and A. Ichikawa, "Attitude control of a helicopter model by robust PID controllers", in *IEEE International Conference on Control Applications*, Munich, 2006, pp. *1971-1976.*

[7] H. J. Kim and D. H. Shim, "A flight control system for aerial robots: algorithms and experiments", *Cont Eng Pract*, vol. *11,* no. *12,* pp. *1389-1400,* Dec., 2003.

[8] J. Wu, G. Huang and Y. Fan, "An attitude control method of unmanned helicopter based on adaptive output feedback", in *3rd International Conference on Intelligent System and Knowledge Engineering*, Xiamen, 2008, pp. *748-753.*

[9] P. Song, G. Qi and K. Li, "The flight control system based on multivariable PID neural network for small-scale unmanned helicopter", in *International Conference on Information Technology and Computer Science*, Kiev, 2009, pp. *538-541.*

[10] R. Enns and J. Si, "Helicopter trimming and tracking control using direct neural dynamic programming", *IEEE T Neural Networ*, vol. *14,* no. *4,* pp. *929-939,* July, 2003.

[11] A. Isidori, L. Marconi and A. Serrani, "Robust nonlinear motion control of a helicopter", *IEEE T Automat Contr*, vol. *48,* no. *3,* pp. *413-426,* Mar., 2003.

[12] M. F. Weilenmann, U. Christen and H. Geering, "Robust helicopter position control at hover ", in *American Control Conference*, Baltimore, 1994, vol. *3*, pp. *2491-2495.*

[13] D. Fujiwara, J. Shin, K. Hazawa and K. Nonami, " H_∞ hovering and guidance control for autonomous small-scale unmanned helicopter", in *IEEE/RSJ International Conference on Intelligent Robots and Systems*, Sendai, 2004,vol. *3*, pp. *2463-2468.*

[14] J. Gadewadikar, F.L. Lewis and B. M. Chen, "Structured H_∞ command and control loop design for unmanned helicopters", *J Guid Control Dynam*, vol. *31,* no. *4,* pp. *1093-1102,* Aug., 2008.

[15] J. Gadewadikar, F.L. Lewis, K. Subbarao, K. Peng and B. M. Chen, " H_∞ static output feedback control for rotorcraft", *J Intell Robot Syst*, vol. *54,* no. *4,* pp. *629-646,* Apr., 2009.

[16]K. Peng, G. Cai, B. M. Chen, M. Dong, K. y. Lum and T. H. Lee, "Design and development of an autonomous flight control law for a UAV helicopter", *Automatica*, vol. *45,* no. *10,* pp. *2333-2338,* Oct., 2009.

[17] S. Tarbouriech, G. Garcia, J. M. Gomes da Silva jr. and I. Queinnec, "Stability and Stabilization of Linear Systems with Saturating Actuators ", Springer, London, 2011.

[19] T. Hu, Z. Lin and B. M. Chen, "An analysis and design method for linear systems subject to actuator saturation and disturbance", *Automatica*, vol. *38,* no. *2,* pp. *351-359,* Feb., 2002.

[19] S. Tarbouriech, G. Garcia and J. M. Gomes da Silva Jr., "Robust stability of uncertain polytopic linear time-delay systems with saturating inputs: an LMI approach", *Comput Electr Eng*, vol. *28,* no. *3,* pp. *157-169,*May, 2002.

[20] F. Blanchini, "Feedback control for linear time invariant systems with state and control bounds in the presence of disturbances", *IEEE T Automat Contr*, vol. *35,* no. *11,* pp. *1231-1234,* Nov.,1990.

[21] S. Boyd, L. El Gahoui, E. Feron. and V. Balakrishnan, "Linear matrix inequalities in system and control theory ", SIAM Studies in Applied Mathematics, Philadelphia, 199.

[22] J. Löfberg, "Yalmip: a toolbox for modelling and optimization in Matlab", in *IEEE International Symposium on Computer Aided Control System Design*, Taipei, 2004, pp. *284-289.*

[23] J. F. Strum, "Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones", *Optim Method Softw*, vol. *11,* no. *1-4,* pp. *625-653,* 1999.

[24] B. Mettler, "Identification modelling and characteristics of miniature rotorcraft", Springer, 2003.

[25] G. Cai, B.M. Chen, K. Peng, M. Dong and T. H. Lee, "Modelling and control system design for a UAV Helicopter", in *14th Mediterranean Conference on Control and Automation*, Ancona, 2006, pp. *1-6.*

[26] D. H. Shim, H. j. Kim and S. Sastry, "Control system design for rotorcraft based unmanned aerial vehicles using time-domain system identification", in *IEEE International Conference on Control Applications*, Anchorage, 2000, pp. *808-813.*