# PERFORMANCE ANALYSIS OF MAXIMUM LIKELIHOOD 3-D SOURCE LOCALIZATION ALGORITHM

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# ABSTRACT

In this paper the performance of the deterministic maximum likelihood 3-D location estimator for the near-field sources is studied based on the derivation of Cramér-Rao bounds. In the derivation, the source signals and unknown parameters are assumed to be deterministic while the noise is Gaussian. Furthermore, some insights into the achievable performance of the deterministic maximum likelihood approach is obtained by numerical evaluation of the Cramér-Rao bounds.

#### 1. INTRODUCTION

In the past, many approaches addressed for the localization of passive sources using an array of sensors operates under far-field assumption so that they can only estimate azimuth (in 1-D) or azimuth and elevation (in 2-D). [1, 2]. In these approaches, waves reaching from sources to sensor arrays are assumed to be in the form of planar wave-front. However, when a source is located close to the array, the waves impinging on it cannot be assumed to be planar. In such cases, the inherent curvature of the waveforms can no longer be neglected. The scenarios taking into consideration of inherent curvature of the waveforms operate under near-field assumption. Near-field localization algorithms have been widely used in speech enhancement using microphone arrays, underwater source localization, ultrasonic imaging, radar, electronic surveillance and seismic expoloration applications. Recently, a total least squares ESPRIT like algorithm, based on the fourth-order cumulants was proposed in [3]. In [4], a high resolution algorithm that uses only second order statistics of the array outputs was developed. Few of the existing works dealt with passive 3-D source localization [3, 4, 5]. This paper deals with a performance of a new 3-D source localization estimator which involves the estimation of spherical coordinates, namely *azimuth*, *elevation and range* [5]. It is based on the deterministic maximum likelihood (DML) criterion which employs the source signals recorded by 2-D array under near-field assumption.

Establishing bounds on the accuracy that can be achieved in estimation is an important goal since it provides benchmarks for evaluating the performance of the actual estimators. In many signal processing problems, Cramer-Rao bound (CRB) is used as a lower bound for the covariance of estimated parameters. We therefore evaluate the performance of the DML algorithm based on the derivation of the CRB. This bound is often taken as a measure of how well an estimator performs. The CRB derived in this paper for the near-field source parameters and source signals assumes that the signals and unknown parameters are deterministic while the noise is Gaussian .

### 2. SYSTEM MODEL

Consider a near-field scenario in which narrowband signals from d sources received by an  $K \times L$  element

antenna array. Let the array center be the phase reference point with index '(0,0)' as depicted in Figure 1. Assuming 2-D rectangular uniform linear array



Figure 1: Near-field scenario with a 2-D linear array

consisting of omnidirectional sensors with interelement spacing  $\Delta$  along each axes, we write the output of the  $(k, l)^{th}$  sensor with narrowband, co-channel signal as,

$$x_{k,l}(t_n) = \sum_{i=1}^d s_i(t_n) e^{j\tau_{kl}(i)} + n_{k,l}(t_n), \ 1 \le t_n \le N$$
(1)

where  $s_i(t_n)$  denotes the complex envelope of the  $i^{th}$  source signal,  $n_{k,l}(t_n)$  is an additive complex Gaussian sensor noise and  $\tau_{kl}(i)$  is the phase difference of the  $i^{th}$  signal collected at sensor (k, l) with respect to the  $i^{th}$  signal collected at reference sensor '(0, 0)'. Due to our narrowband assumption, the phase difference is given by

$$\tau_{kl}(i) = \frac{2\pi}{\lambda} (r_{kl}(i) - r_i) \tag{2}$$

where  $\lambda$  is the wavelength of the source wavefronts. The distance between  $i^{th}$  source and the  $(k, l)^{th}$  sensor equals

$$r_{kl}(i) = r_i \sqrt{1 + \frac{(k^2 + l^2)\Delta^2}{r_i^2} - \frac{2\Delta}{r_i}} \sin \theta_i (k \cos \varphi_i + l \sin \varphi_i) (3)$$

The plane wave approximation of far-field sources is obtained by retaining only term up to the first power of  $\frac{\Delta}{r_i}$  and its multiplier in the binomial expansion of (3). Since the near field sources are of interest in this paper, we should include an extra term to approximate the effect of spherical waves. Such an approximation can therefore be obtained by retaining terms up to the second power of  $\frac{\Delta}{r_i}$  and its multiplier in the binomial expansion of (3). We then arrive at the *fresnel* approximation of the distance:

$$r_{kl}(i) \approx r_i - k\Delta \sin \theta_i \cos \varphi_i + \frac{(k^2)\Delta^2}{2r_i} (1 - \sin^2 \theta_i \cos^2 \varphi_i) - l\Delta \sin \theta_i \sin \varphi_i + \frac{l^2 \Delta^2}{2r_i} (1 - \sin^2 \theta_i \sin^2 \varphi_i) - \frac{kl\Delta^2}{2r_i} \sin^2 \theta_i \sin 2\varphi_i) .$$
(4)

The phase difference  $\tau_{kl}(i) = [\omega_{xi}k + \phi_{xi}k^2 + \omega_{yi}l + \phi_{yi}l^2 + \beta_ikl]$  is then

$$\tau_{kl}(i) \approx \left(\frac{2\pi\Delta}{\lambda}\sin\theta_i\cos\varphi_i\right)k - \left(\frac{\pi\Delta^2}{\lambda_i}(1-\sin^2\theta_i\cos^2\varphi_i)\right)k^2 \\ + \left(\frac{2\pi\Delta}{\lambda}\sin\theta_i\sin\varphi_i\right)l - \left(\frac{\pi\Delta^2}{\lambda r_i}(1-\sin^2\theta_i\sin^2\varphi_i)\right)l^2 \\ + \left(\frac{\pi\Delta^2}{\lambda r_i}\sin^2\theta_i\sin2\varphi_i\right)kl$$
(5)

where the parameters  $\omega_{xi}$ ,  $\phi_{xi}$ ,  $\omega_{yi}$ ,  $\phi_{yi}$  and  $\beta i$  are nonlinear functions of the azimuth  $\theta_i$ , elevation  $\varphi_i$ and range  $r_i$  of the  $i^{th}$  source:

$$\omega_{xi} = \frac{2\pi\Delta}{\lambda}\sin\theta_i\cos\varphi_i, \quad \phi_{xi} = -\frac{\pi\Delta^2}{\lambda r_i}(1-\sin^2\theta_i\cos^2\varphi_i),$$
$$\omega_{yi} = \frac{2\pi\Delta}{\lambda}\sin\theta_i\sin\varphi_i, \quad \phi_{yi} = -\frac{\pi\Delta^2}{\lambda r_i}(1-\sin^2\theta_i\sin^2\varphi_i),$$
$$\beta i = \frac{\pi\Delta^2}{\lambda r_i}\sin^2\theta_i\sin2\varphi_i. \tag{6}$$

Then, the noise corrupted array measurements at the  $(k, l)^{th}$  sensor can be approximately expressed as:

$$x_{k,l}(t_n) = \sum_{i=1}^d s_i(t_n) e^{j[\omega_{xi}k + \phi_{xi}k^2 + \omega_{yi}l + \phi_{yi}l^2 + \beta_i kl]} + n_{k,l}(t_n).$$
(7)

For a collection of observed outputs of  $K \times L$  sensors in 2-D array  $\mathbf{x}(t_n) = [\mathbf{x}^T_{L_{min}}(t_n), \cdots, \mathbf{x}^T_{L_{max}}(t_n)]^T$ , the matrix formulation of (1) is obtained as follows

$$\mathbf{x}(t_n) = \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\varphi}, \mathbf{r})\mathbf{s}(t_n) + \mathbf{n}(t_n), \quad 1 \le t_n \le N \quad (8)$$

where the super vector  $\mathbf{x}(t_n)$  consists of  $\mathbf{x}_l(t_n) = [x_{K_{min,l}}(t_n) \cdots x_{K_{max,l}}(t_n)]^T$  which is only one column array vector,  $\mathbf{s}(t_n) = [s_1(t_n) \cdots s_d(t_n)]^T$  is the collection of d source signals impinging to 2-D array,  $\mathbf{n}(t_n) = [\mathbf{n}_{L_{min}}^T(t_n) \cdots \mathbf{n}_{L_{max}}^T(t_n)]^T$  is super Gaussian complex vector with zero-mean and known spatial covariance  $\sigma^2 \mathbf{I}$ , which consists of column array vectors one forming as  $\mathbf{n}_l(t_n) = [\mathbf{n}_{K_{min,l}}(t_n) \cdots \mathbf{n}_{K_{max,l}}(t_n)]^T$ ,  $\mathbf{A}(\theta, \varphi, \mathbf{r}) = [\mathcal{A}_1(\theta, \varphi, \mathbf{r}) \cdots \mathcal{A}_d(\theta, \varphi, \mathbf{r})]$  is the arrays steering matrix in the near-field scenario which is known as a function of unknown set of parameters  $\{\theta, \varphi, \mathbf{r}\} = \{(\theta, \varphi, \mathbf{r})_1 \cdots (\theta, \varphi, \mathbf{r})_d\}$ , consisting of column array steering vectors one forming as  $\mathcal{A}_i(\theta, \varphi, \mathbf{r}) = [\mathbf{a}_{L_{min}}^T(\theta, \varphi, \mathbf{r}) \cdots \mathbf{a}_{L_{max}}^T(\theta, \varphi, \mathbf{r})]^T$  and  $\mathbf{a}_l(\theta, \varphi, \mathbf{r})$  is  $l^{th}$  column array steering vector for  $i^{th}$  source, in the following form

$$\mathbf{a}_{l}(\boldsymbol{\theta}, \boldsymbol{\varphi}, \mathbf{r}) = \begin{bmatrix} e^{j\tau_{K_{min}l}(i)} \\ \vdots \\ 1 \\ e^{j\tau_{1l}(i)} \\ e^{j\tau_{2l}(i)} \\ \vdots \\ e^{j\tau_{K_{max}l}(i)} \end{bmatrix} .$$
(9)

The joint azimuth, elevation and range estimation problem is, for given array observations, to find the azimuth  $\boldsymbol{\theta}$ , elevation  $\boldsymbol{\varphi}$ , and range **r** using the model (8). Many of the well known methods such as ML, subspace fitting, and MUSIC that have been developed for DOA model are applicable to the 3-D source localization problem. ML algorithms estimate the desired parameters by solving an optimization problem of the general form

$$\left[\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{r}}, \hat{\mathbf{s}}\right] = \arg \min_{\boldsymbol{\theta}, \boldsymbol{\varphi}, \mathbf{r}} \sum_{t_n=1}^{N} \|\mathbf{x}(t_n) - \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\varphi}, \mathbf{r}) \mathbf{s}(t_n)\|^2 (10)$$

For the solution to (10) to be ML, however, requires additional conditions to be satisfied: first that the noise  $\mathbf{n}(t_n)$  is Gaussian and second, the source signals are either deterministic or stochastic.

# 3. MAXIMUM LIKELIHOOD ESTIMATION

We focus on the DML approach performance in the sequel. In DML case we do not make any statistical assumption on the source signal and treat it as unknown but deterministic quantity. Since the noise vector  $\boldsymbol{n}(t_n)$  is assumed to be additive, Gaussian with covariance matrix  $\sigma^2 \mathbf{I}$ , the negative log-likelihood function (after neglecting unnecessary terms) can be written as

$$\mathcal{L}(\boldsymbol{x};\boldsymbol{\theta},\boldsymbol{\varphi},\mathbf{r},\boldsymbol{s}) = -\sum_{t_n=1}^{N} \left[\mathbf{x}(t_n) - \mathbf{A}(\boldsymbol{\theta},\boldsymbol{\varphi},\mathbf{r})\mathbf{s}(t_n)\right]^{H} \times \left[\mathbf{x}(t_n) - \mathbf{A}(\boldsymbol{\theta},\boldsymbol{\varphi},\mathbf{r})\mathbf{s}(t_n)\right]$$
(11)

The ML estimate of the parameters  $\hat{\boldsymbol{s}}$  and  $\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{r}}\}$  is a choice of parameters  $\boldsymbol{s}$  and  $\{\boldsymbol{\theta}, \boldsymbol{\varphi}, \mathbf{r}\}$  which locally maximizes the log-likelihood function (11). Typically the maximization problem of (11) is solved in two steps. First, maximize (11) with respect to  $\mathbf{s}(t_n)$ , keeping  $\{\theta, \varphi, \mathbf{r}\}$  fixed. The first step results in closed form solution, since the maximization with respect to  $\mathbf{s}(t_n)$  is a linear least-squares problem. However, the second step, maximization of the log-likelihood function with respect to  $\{\theta, \varphi, \mathbf{r}\}$ , results in a complicated multiparameter optimization problem and does not yield to a closed form solution. Solutions of such problems usually requires numerical methods, such as the methods of Scoring, Newton-Raphson or some other gradient search algorithm. However, for the problem at hand, these numerical methods tend to be computationally complex. Therefore, Expectation/Maximization based iterative approach is proposed in [5]. In this paper, we evaluate the performance of that proposed DML estimator in the sequel.

#### 4. CRAMER-RAO BOUNDS

The performance of the deterministic ML method is evaluated based on the derivation of CRB for the unbiased estimates of the nonrandom parameters. To prove the deterministic CRB, let the parameter vector be defined as  $\boldsymbol{\eta} \in \mathbf{C}^{(N+3)d \times 1}$ 

$$\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{s}^T \ \boldsymbol{\theta}^T \ \boldsymbol{\varphi}^T \ \mathbf{r}^T \end{bmatrix}^T$$
(12)

where  $\boldsymbol{s} \in \mathbf{C}^{Nd \times 1}$  is the source signals vector,  $\boldsymbol{\theta} \in \mathbf{R}^{d \times 1}$ ,  $\boldsymbol{\varphi} \in \mathbf{R}^{d \times 1}$  and  $\mathbf{r} \in \mathbf{R}^{d \times 1}$  are the near-field source location parameter vectors.

The *CRB* provides a lower bound on the variance of any unbiased estimator. We derive the lower bound on the covariance matrix of  $\hat{\eta}$  for the 3-D near-field localization problem. Suppose  $\hat{\eta}$  is an unbiased estimator of a vector of deterministic unknown parameters  $\eta$  (i.e.,  $E\{\hat{\eta}\} = \eta$ ) then the estimator's covariance matrix satisfies

$$\mathbf{J}^{-1}(\boldsymbol{\eta}) \le E\left\{(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}})(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}})^T\right\}$$
(13)

where  $\mathbf{J}(\boldsymbol{\eta})$  is the Fisher information matrix (FIM) defined by

$$\mathbf{J}(\boldsymbol{\eta}) = E\left\{ \left(\frac{\partial \mathcal{L}(\boldsymbol{x};\boldsymbol{\eta})}{\partial \boldsymbol{\eta}}\right) \left(\frac{\partial \mathcal{L}(\boldsymbol{x};\boldsymbol{\eta})}{\partial \boldsymbol{\eta}}\right)^T \right\} .$$
 (14)

## 4.1. Computation of the Derivatives

We now start constructing the FIM by computing the derivative of (11) with respect to  $\boldsymbol{\eta}' = [\mathbf{s}_r^T(1) \mathbf{s}_c^T(1) \cdots \mathbf{s}_r^T(N) \mathbf{s}_c^T(N) \boldsymbol{\theta}^T \boldsymbol{\varphi}^T \mathbf{r}^T]^T$  where

$$\mathbf{s}_{r}(t_{n}) = \operatorname{Re}\{[s_{1}(t_{n}), \cdots, s_{d}(t_{n})]^{T}\}$$
(15)  
$$\mathbf{s}_{c}(t_{n}) = \operatorname{Im}\{[s_{1}(t_{n}), \cdots, s_{d}(t_{n})]^{T}\}$$
$$\boldsymbol{\theta} = [\theta_{1}, \cdots, \theta_{d}]^{T}$$
$$\boldsymbol{\varphi} = [\varphi_{1}, \cdots, \varphi_{d}]^{T}$$
$$\mathbf{r} = [r_{1}, \cdots, r_{d}]^{T}$$
$$\boldsymbol{\tau} = [\boldsymbol{\theta}^{T} \ \boldsymbol{\varphi}^{T} \ \mathbf{r}^{T}]^{T} .$$

For notational simplicity, we omitted  $(\boldsymbol{\theta}, \boldsymbol{\varphi}, \mathbf{r})$  in the sequel. The partial derivatives of  $\mathcal{L}(\boldsymbol{x}; \boldsymbol{\eta}')$  with respect to  $\boldsymbol{\eta}'$  are:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{s}_{r}(t_{n})} = \frac{2}{\sigma^{2}} \operatorname{Re} \left\{ \mathbf{A}^{H} \mathbf{n}(t_{n}) \right\} \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{s}_{c}(t_{n})} = \frac{2}{\sigma^{2}} \operatorname{Im} \left\{ \mathbf{A}^{H} \mathbf{n}(t_{n}) \right\}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_{i}} = \frac{2}{\sigma^{2}} \sum_{t_{n}=1}^{N} \operatorname{Re} \left\{ s_{i}^{*}(t_{n}) \frac{\partial \mathcal{A}_{i}^{H}}{\partial \theta_{i}} \mathbf{n}(t_{n}) \right\}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_{i}} = \frac{2}{\sigma^{2}} \sum_{t_{n}=1}^{N} \operatorname{Re} \left\{ s_{i}^{*}(t_{n}) \frac{\partial \mathcal{A}_{i}^{H}}{\partial \varphi_{i}} \mathbf{n}(t_{n}) \right\}$$

$$\frac{\partial \mathcal{L}}{\partial r_{i}} = \frac{2}{\sigma^{2}} \sum_{t_{n}=1}^{N} \operatorname{Re} \left\{ s_{i}^{*}(t_{n}) \frac{\partial \mathcal{A}_{i}^{H}}{\partial \varphi_{i}} \mathbf{n}(t_{n}) \right\}$$

The partial derivatives of the log-likelihood function with respect to the near-field parameters written more compactly,

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\tau}} = \frac{2}{\sigma^2} \sum_{t_n=1}^{N} \operatorname{Re} \left\{ \mathcal{S}^H(t_n) \mathbf{D}_{\boldsymbol{\tau}}^H \mathbf{n}(t_n) \right\}$$
(17)

where

$$\mathbf{S}(t_n) = diag[s_1(t_n) \cdots s_d(t_n)]$$
(18)  
$$\mathcal{S}(t_n) = \begin{bmatrix} \mathbf{S}(t_n) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}(t_n) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}(t_n) \end{bmatrix}$$
$$\mathbf{D}_{\boldsymbol{\tau}} = \begin{bmatrix} \frac{\partial \mathcal{A}(\boldsymbol{\theta}_1, \boldsymbol{\varphi}_1, \boldsymbol{r}_1)}{\partial \theta_1}, \cdots, \frac{\partial \mathcal{A}(\boldsymbol{\theta}_d, \boldsymbol{\varphi}_d \boldsymbol{r}_d)}{\partial \theta_d}; \frac{\partial \mathcal{A}(\boldsymbol{\theta}_1, \boldsymbol{\varphi}_1, \boldsymbol{r}_1)}{\partial \varphi_1}, \\ \cdots, \frac{\partial \mathcal{A}(\boldsymbol{\theta}_d, \boldsymbol{\varphi}_d, \boldsymbol{r}_d)}{\partial \varphi_d}; \frac{\partial \mathcal{A}(\boldsymbol{\theta}_1, \boldsymbol{\varphi}_1, \boldsymbol{r}_1)}{\partial r_1}, \cdots, \frac{\partial \mathcal{A}(\boldsymbol{\theta}_d, \boldsymbol{\varphi}_d, \boldsymbol{r}_d)}{\partial r_d} \end{bmatrix}.$$

# 4.2. Evaluation of the FIM Matrix

We need the following assumption and results to further proceed, (see e.g., [1]):

$$E[\mathbf{n}(t_n)\mathbf{n}^H(t_m)] = \sigma^2 \mathbf{I}$$
(19)  

$$E[\mathbf{n}(t_n)\mathbf{n}^T(t_m)] = 0$$
  

$$E[\mathbf{n}^H(t_n)\mathbf{n}(t_n)\mathbf{n}^T(t_m)] = 0 .$$

Using the assumption and results given above, the elements of the information matrix can be obtained as

$$E\left\{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{s}_{r}(t_{n})}\right)\left(\frac{\partial \mathcal{L}}{\partial \mathbf{s}_{r}(t_{m})}\right)^{T}\right\} = \frac{2}{\sigma^{2}}\operatorname{Re}\left\{\mathbf{A}^{H}\mathbf{A}\right\}\delta_{n,m}$$

$$E\left\{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{s}_{r}(t_{n})}\right)\left(\frac{\partial \mathcal{L}}{\partial \mathbf{s}_{c}(t_{m})}\right)^{T}\right\} = -\frac{2}{\sigma^{2}}\operatorname{Im}\left\{\mathbf{A}^{H}\mathbf{A}\right\}\delta_{n,m}$$

$$E\left\{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{s}_{c}(t_{n})}\right)\left(\frac{\partial \mathcal{L}}{\partial \mathbf{s}_{c}(t_{m})}\right)^{T}\right\} = \frac{2}{\sigma^{2}}\operatorname{Re}\left\{\mathbf{A}^{H}\mathbf{A}\right\}\delta_{n,m}$$

$$E\left\{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{s}_{r}(t_{n})}\right)\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\tau}}\right)^{T}\right\} = \frac{2}{\sigma^{2}}\operatorname{Re}\left\{\mathbf{A}^{H}\mathbf{D}_{\boldsymbol{\tau}}\mathcal{S}(t_{n})\right\}$$

$$E\left\{\left(\frac{\partial \mathcal{L}}{\partial \mathbf{s}_{c}(t_{n})}\right)\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\tau}}\right)^{T}\right\} = \frac{2}{\sigma^{2}}\operatorname{Im}\left\{\mathbf{A}^{H}\mathbf{D}_{\boldsymbol{\tau}}\mathcal{S}(t_{n})\right\}$$

$$E\left\{\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\tau}}\right)\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\tau}}\right)^{T}\right\} = \frac{2}{\sigma^{2}}\operatorname{Im}\left\{\mathbf{A}^{H}\mathbf{D}_{\boldsymbol{\tau}}\mathcal{S}(t_{n})\right\}$$

# 4.3. Bounds for the Near-Field Parameters

To obtain the bounds only for the near-field parameters, we will use the partitioned matrix inversion lemma. Let us then introduce following notation to form partitioned FIM :

$$\mathcal{A}_{r} = \frac{2}{\sigma^{2}} \operatorname{Re} \left\{ \mathbf{A}^{H} \mathbf{A} \right\}$$
(20)

$$\mathcal{A}_{c} = \frac{2}{\sigma^{2}} \operatorname{Im} \left\{ \mathbf{A}^{H} \mathbf{A} \right\}$$
$$\mathbf{\Lambda}_{r}(t_{n}) = \frac{2}{\sigma^{2}} \operatorname{Re} \left\{ \mathbf{A}^{H} \mathbf{D}_{\tau} \mathcal{S}(t_{n}) \right\}$$
$$\mathbf{\Lambda}_{c}(t_{n}) = \frac{2}{\sigma^{2}} \operatorname{Im} \left\{ \mathbf{A}^{H} \mathbf{D}_{\tau} \mathcal{S}(t_{n}) \right\}$$
$$\mathbf{J}(\tau) = \frac{2}{\sigma^{2}} \sum_{t_{n}=1}^{N} \operatorname{Re} \left\{ \mathcal{S}^{H}(t_{n}) \mathbf{D}_{\tau}^{H} \mathbf{D}_{\tau} \mathcal{S}(t_{n}) \right\}$$

Then the FIM can be written in partitioned form as

$$\mathbf{J}(\boldsymbol{\eta}') = \begin{bmatrix} \mathcal{A}_r & -\mathcal{A}_c & \mathbf{0} & \mathbf{\Lambda}_r(1) \\ \mathcal{A}_c & \mathcal{A}_r & & \mathbf{\Lambda}_c(1) \\ & \ddots & & \vdots \\ & & \mathcal{A}_r & -\mathcal{A}_c & \mathbf{\Lambda}_r(N) \\ \mathbf{0} & & \mathcal{A}_c & \mathcal{A}_r & \mathbf{\Lambda}_c(N) \\ \hline \mathbf{\Lambda}_r^T(1) & \mathbf{\Lambda}_c^T(1) & \cdots & \mathbf{\Lambda}_r^T(N) & \mathbf{\Lambda}_c^T(N) & \mathbf{J}(\boldsymbol{\tau}) \end{bmatrix}$$
(21)

If we employ a standard result on the inverse of the partitioned matrix, we obtain

$$CRB^{-1}(\boldsymbol{\tau}) = \mathbf{J}(\boldsymbol{\tau}) - \mathbf{\Lambda}^{T} \begin{bmatrix} \mathcal{A}_{r} & -\mathcal{A}_{c} & \mathbf{0} \\ \mathcal{A}_{c} & \mathcal{A}_{r} & & \\ & \ddots & & \\ \mathbf{0} & & \mathcal{A}_{c} & \mathcal{A}_{r} \end{bmatrix}^{-1} \mathbf{\Lambda}$$

$$(22)$$

where  $\mathbf{\Lambda} = [\mathbf{\Lambda}_r^T(1) \ \mathbf{\Lambda}_c^T(1) \ \cdots \ \mathbf{\Lambda}_r^T(N) \ \mathbf{\Lambda}_c^T(N)]^T$ . Finally, we obtain *CRB* for the parameters of interest as

$$CRB^{-1}(\boldsymbol{\tau}) = \frac{2}{\sigma^2} \sum_{t_n=1}^{N} \operatorname{Re} \left\{ \mathcal{S}^{H}(t_n) \mathbf{D}_{\boldsymbol{\tau}}^{H} \right\}$$

$$\times \left[ \mathbf{I} - \mathbf{A} (\mathbf{A}^{H} \mathbf{A})^{-1} \mathbf{A}^{H} \right] \mathbf{D}_{\boldsymbol{\tau}} \mathcal{S}(t_n)$$
(23)

A more explicit individual CRB expressions for the near-field parameters  $\boldsymbol{\theta}$ ,  $\boldsymbol{\varphi}$  and  $\mathbf{r}$  can be obtained by using a result on the partitioned matrix and its inverse. We then have

$$CRB^{-1}(\boldsymbol{\theta}, \boldsymbol{\varphi}) = \boldsymbol{\mu} - \boldsymbol{\alpha} \boldsymbol{H}^{-1} \boldsymbol{\alpha}^{T} = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{V} \\ \boldsymbol{V}^{T} & \boldsymbol{U} \end{bmatrix},$$
  

$$CRB^{-1}(\boldsymbol{\theta}) = \boldsymbol{T}, CRB^{-1}(\boldsymbol{\varphi}) = \boldsymbol{U},$$
  

$$CRB^{-1}(\mathbf{r}) = \boldsymbol{H} - \boldsymbol{\alpha}^{T} \boldsymbol{\mu}^{-1} \boldsymbol{\alpha},$$
(24)

where

$$CRB^{-1}(\boldsymbol{\tau}) = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} & \boldsymbol{E} \\ \boldsymbol{D}^T & \boldsymbol{F} & \boldsymbol{G} \\ \boldsymbol{E}^T & \boldsymbol{G}^T & \boldsymbol{H} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\alpha} \\ \boldsymbol{\alpha}^T & \boldsymbol{H} \end{bmatrix} .$$
(25)

# 5. SIMULATIONS

A 2 – D Uniform linear array of K = L = 3 sensors with inter-element spacing  $\Delta = \frac{\lambda}{2}$  was used to estimate the locations of two sources located at  $\{\theta_1, \varphi_1, r_1\} =$   $\{-65^0, -20^0, 2.5\lambda\}$  and  $\{\theta_2, \varphi_2, r_2\} = \{45^0, -30^0, 4\lambda\}$ . The number of the snapshots (N) was set to 500.

We tested the proposed method for different signal to noise ratios (SNR = 0 - 30dB and K = 200 trials per each SNR point). In each trial, the RMSE of the estimations for  $\{\theta_1, \varphi_1, r_1\}$  and  $\{\theta_2, \varphi_2, r_2\}$  were recorded and the corresponding results are presented only for source 1 due to lack of space in the Figure 2, Figure 3, and Figure 4 respectively. The theoretical Cramer-Rao Bound results were compared with the experimental DML estimator performance results.

#### 6. CONCLUSIONS

In this paper, we derived lower bound on the covariance matrix of the proposed unbiased estimator and presented Monte Carlo simulations to verify the theoretically predicted estimator's performance. The examples demonstrated that the deterministic ML algorithm achieve the CRB for high SNR values.



Figure 2: CRB and RMS error for elevation



Figure 3: CRB and RMS error for azimuth



Figure 4: CRB and RMS error for range

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