An Application of GBS Method to the Design of Non-Linear Controller for Continues & Autonomous Systems

Ali Akbar Gharaveisi M. Rashidi-Njad M. A. Sadrnia

a_gharaveisi@yahoo.com masadrnia@yahoo.com mrashidi@mail.uk.ac.ir

Shahrood University of Technology, Shahrood, Iran

International Research Center of High-Tech & Environmental Sciences, Mahan, Iran

Shahid Bahonar University of Kerman, Kerman, Iran

Keywords: Generalized Back-stepping, Non-Linear Control, Strict Feedback

ABSTRACT

In this paper a new design methodology for a wide range of non-linear control systems via the so-called generalized back-stepping(GBS) technique is proposed. The new proposed criterion can be implemented to tracking problem as well as stability of non-linear autonomous control systems. This model that can be called "feedback state of non-linear control systems" may cover most of the non-linear control systems. By using the suggested model, unstable state of stabled system as well as tracking problem can be derived. Case study and simulation results show the effectiveness of the proposed technique.

1. Introduction

Lyapunov theory is a well-known proper mean for linear and non-linear systems analysis. The major problem of that theory, which can be pointed, especially for non-linear systems is to derive a function such that it should satisfy the Lyapunov conditions. If such a function is derived, system stability can be guaranteed, while in this regard the designer's experiences are also desired. Although regarding this issue, there are several proposed methods available while each individual may face with some particular constraints. Some general methods to determine the Lyapunov functions are:

1-Method of linearization around the operating point; where the major issue for this technique is eliminating the non-linear dynamics of system as well as procuring local stability.

2-Crossofskey method; where in the case of large number of system states, solving the related equations and determining of conditions can be a tough job.

3-Generalized Crosophskey method; where in this method determining of conditions are easy job, while computational works are so high.

4-Variable gradient method; while in this method solving the equations is not so easy; whereas the results are similar to the method of linearization.

Regarding to the above issues; our attempts is ended to a simple proposed technique the so-called back-stepping methodology. This technique is a backward technique that can help one to find the Lyapunov functions. One of the advantages of this method is to prevent eliminating nonlinear dynamics of the system. In fact, back-stepping method is a modification from state feedback of linear systems to non-linear systems by using Lyapunov theories. It seems that the origin of back-stepping theory is not precisely recognized, while some concurrent analyses with regards to this method has been done. The most important study from the literature can be addressed to some research papers of the 1980 decade. It is important to mention that the researches of Kokotowich and his colleagues have introduced this issue [1]. In 1991 Kokotowich et. al. presented this idea through his published paper [2]. Kanlacupulos proposed a mathematical for designing a non-linear controller using back-stepping technique [3]. Follow to these researches some years later, researchers such as Christic [4], Freeman [5] and Spultcher [6] published several research paper with regards to this subject. Also Kokotowich in 1999 at international IFAC symposium reviewed the progresses of back-stepping technique during 1990 decade [7]. In the following this method will be discussed id details.

2. Back-Stepping Technique

This method can be applied to some particular models of non-linear systems, the so-called explicit feedback systems. These systems can be presented using the following mathematical relationships:

$$\begin{cases} \dot{z}_1 = f_1(z_1) + g_1(z_1) z_2 \\ \dot{z}_2 = f_2(z_1, z_2) + g_2(z_1, z_2) z_3 \\ \vdots \qquad \vdots \\ \dot{z}_{n-1} = f_{n-1}(z_1, \dots, z_{n-1}) + g_{n-1}(z_1, \dots, z_{n-1}) z_n \\ \dot{z}_n = f_n(z_1, \dots, z_n) + g_n(z_1, \dots, z_n) u \end{cases} \quad (1)$$

In [8] by using (1) in order to derive the Lyapunov functions as well as a control signal *u,* a backward algorithm is implemented. In that algorithm at first stage by assuming Lyapunov functions as $V(z_1)$ and control signal as z_2 for first term, Lyapunov function and control signal for two other states z_1 و z_2 can be derived. In next stage z_3 is introduced as control signal, while it can be calculated based upon the previous stated and Lyapunov function will be derived. These calculation sequences will be continuing until the final stage, reaching to Lyapunov function of whole system as well as control signal *u.*

Back-stepping technique has some weakness as the following:

- It has restrictions, while is only applicable for particular non-linear systems that obey equations (1).
- Using this method for *n* states system *n-1* backward iteration is needed to be done, where lots of computational jobs are required.

In next part a generalized back-stepping technique will be discussed, where Lyapunov function as well as control signal will be derived in one stage.

3. GBS Technique

A specific class of autonomous non-linear systems and continues of k order can be modeled as:

$$\begin{cases} \dot{X} = F(X) + G(X)\eta \\ \dot{\eta} = f_0(X,\eta) + g_0(X,\eta) u \end{cases} \quad (2)$$

Where : $X = [x_1, x_2, \dots, x_{n-1}] \in \Re^{n-1}$, $\eta \in \Re$. Most of non-linear control systems either can be presented by Eq. (2) (e.g. auto-piloting system) or they can be converted to this model easily (e.g. Lorentz equations).

Now by using mathematical model (2), for determining Lyapunov function and control signal the following theorem will be developed.

Theorem: assume that a non-linear control system is presented by Eq. (2). Then a scalar function $V(X)$ is defined as:

$$V(x_1, x_2, \cdots, x_{n-1}) = \frac{1}{2}\sum_{i=1}^{n-1} x_i^2 \quad (3)$$

For the system of Eq. (2), a scalar function $\eta = \Phi_i(x_1, x_2, \cdots, x_{n-1})$, $i = 1,2,\cdots,n-1$, will be defined such that $\Phi_i(0) = 0$ and the function $V(X)$ of Eq. (3) be positive definite, while its derivative also be negative definite. The stabilizer control signal and Lyapunov of whole system can be shown by the following mathematical formula:

$$u = \frac{1}{g_0(X,\eta)}\Big\{\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\frac{\partial \Phi_i}{\partial x_j}[f_j(X) + g_j(X)\eta] - \sum_{i=1}^{n-1} x_i\, g_i(X) - \sum_{i=1}^{n-1} k_i\,[\eta - \Phi_i(X)] - f_0(X,\eta)\Big\} \quad (4)$$

$$\forall\ k_i > 0 \ ; \ i = 1,2,\dots,n-1$$

$$V_t(X,\eta) = \frac{1}{2}\sum_{i=1}^{n-1} x_i^2 + \frac{1}{2}\sum_{i=1}^{n-1}[\eta - \Phi_i(X)]^2 \quad (5)$$

Where : $F = [f_1, f_2, \cdots, f_{n-1}]^T$, $G = [g_1, g_2, \cdots, g_{n-1}]^T$. It can be mentioned that stability of controlled system is of the globally asymptotic stability (GAS) type.

Proof: Eq. (2)can be expended to the following:

$$\begin{cases} \dot{x}_i = f_i(X) + g_i(X)\eta \ ; \ i = 1,2,\dots,n-1 \\ \dot{\eta} = f_0(X,\eta) + g_0(X,\eta) u \end{cases} \quad (6)$$

Regarding the above assumption in which $V(X)$ is positive definite it can be written:

$$\dot{V}(X) = \sum_{i=1}^{n-1} x_i \dot{x}_i = \sum_{i=1}^{n-1} x_i[f_i(X) + g_i(X)\Phi_i(X)] \quad (7)$$

Then, we can have $\Rightarrow \quad \dot{V}(X) \le -W(X)$.

Where $W(X)$ is a positive definite function. By substituting $u_0 = f_0(X,\eta) + g_0(X,\eta)u$ and adding/ subtracting $g_i(X)\Phi_i(X)$ to the *i-th* column of Eq. (6) it can be derived:

$$\begin{cases} \dot{x}_i = [f_i(X) + g_i(X)\Phi_i(X)] + g_i(X)[\eta - \Phi_i(X)] \\ \dot{\eta} = u_0 \qquad , \qquad i = 1,2,\dots,n-1 \end{cases} \quad (8)$$

Now it can be rewritten such that:

$$z_i = \eta - \Phi_i(X) \Rightarrow \dot{z}_i = u_0 - \dot{\Phi}_i(X) \quad (9)$$

$$\dot{\Phi}_i(X) = \sum_{j=1}^{n-1} \frac{\partial \Phi_i}{\partial x_j}[f_j(X) + g_j(X)\eta] \quad (10)$$

And then Eq. (8) can be shown by:

$$\begin{cases} \dot{x}_i = [f_i(X) + g_i(X)\Phi_i(X)] + g_i(X)[\eta - \Phi_i(X)] \\ \dot{z}_i = u_0 - \dot{\Phi}_i \qquad , \qquad i = 1,2,\dots,n-1 \end{cases} \quad (11)$$

In fact, it is known that all z_i have (n-1) terms, where all λ_i can be assumed of (n-1) terms as well:

$$\lambda_i = u_o - \dot{\Phi}_i \quad , \quad i = 1,2,\cdots,n-1 \quad (12)$$

And then :

$$\dot{z}_i = \lambda_i \quad , \quad i = 1,2,\cdots,n-1 \quad (13)$$

Now it is proven that the following relationship can be a Lyapunov function of system. (2):

$$V_t(X,\eta) = \frac{1}{2}\sum_{i=1}^{n-1} x_i^2 + \frac{1}{2}\sum_{i=1}^{n-1}[\eta - \Phi_i(X)]^2 = \frac{1}{2}\sum_{i=1}^{n-1} x_i^2 + \frac{1}{2}\sum_{i=1}^{n-1} z_i^2 \quad (14)$$

It can be seen that the defined function of Eq. (14) is a positive definite function. It is needed to check the first derivative of the function where it is negative definite. It is done in the following:

$$\dot{V}_t(X,\eta) = \sum_{i=1}^{n-1}\frac{\partial V(X)}{\partial x_i}[f_i(X) + g_i(X)\Phi_i(X)] + \sum_{i=1}^{n-1}\frac{\partial V(X)}{\partial x_i}g_i(X) + \sum_{i=1}^{n-1} z_i \lambda_i \quad (15)$$

In order $\dot{V}_t(X,\eta)$ be negative definite, λ_i can be assumed as:

$$\lambda_i = -\frac{\partial V(X)}{\partial x_i}g_i(X) - k_i z_i \quad ; \quad \forall k_i > 0 \quad (16)$$

Then it can be pointed out:

$$\dot{V}(X,\eta) = \sum_{i=1}^{n-1} x_i [f_i(X) + g_i(X)\Phi_i(X)] - \sum_{i=1}^{n-1} k_i z_i^2 \le -W(X) - \sum_{i=1}^{n-1} k_i z_i^2 \quad (17)$$

Eq. (17) satisfies that $\dot{V}_t(X,\eta)$ is negative definite, which confirm the Eq. (14) is a Lyapunov function for system Eq. (2). For the control signal u_0 that stabilizes the system, by using equations (8), (10), (11), it can be written as:

$$u_0 = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\frac{\partial \Phi_i}{\partial x_j}[f_j(X) + g_j(X)\eta] - \sum_{i=1}^{n-1} x_i g_i(X) - \sum_{i=1}^{n-1} k_i [\eta - \Phi_i(X)] \quad (18)$$

And finally by substituting $u = \dfrac{u_0 - f_0(X,\eta)}{g_0(X,\eta)}$ the control signal can be derived based upon Eq. (4).

Since the region of being positive and negative definite of $\dot{V}_t(X,\eta)$ is the whole state region and $V_t(X,\eta)$ is radially unbounded, then GBA will be guaranteed.

4. Simulation Results & Analysis

The following system is selected to study and analyze the proposed method:

$$\begin{cases} \dot{x}_1 = x_1^3 + x_1 x_2 - x_2 + x_1\eta \\ \dot{x}_2 = x_2^3 + x_2\eta \\ \dot{\eta} = u \end{cases} \quad (19)$$

Equation (19) is a practical example of Eq. (2). As figures (1),(2) and (3) show the states of the system are unstable and after a short period of time they converge to infinite.

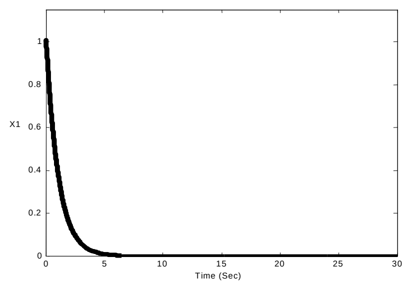

Figure 1- Variations of X_1 Before Stabilization

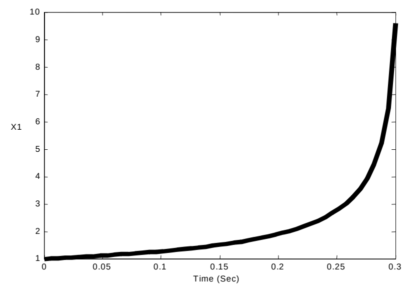

Figure 2- Variations of X_2 Before Stabilization

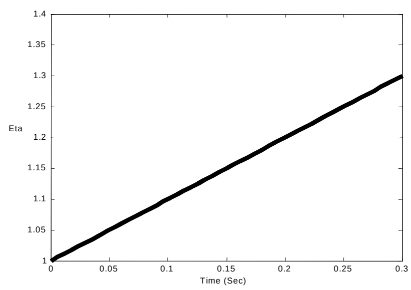

Figure 3- Variations of η Before Stabilization

By comparing Eq. (19) and Eq. (2) of the system it can be written as:

$$\begin{aligned} g_1(X) &= x_1 \quad \Rightarrow \quad g_1(X) \neq 0 \\ f_1(X) &= x_1^3 + x_1 x_2 - x_2 \quad \Rightarrow \quad f_1(X) \neq f_1(x_1) \end{aligned} \quad (20)$$

It can be resulted from Eq. (20) that the system with Eq. (19) is different from Eq. (2). Then it can be said that by using back-stepping method, stabilizing as well as tracking issues of system Eq. (19) is not achievable. In the next section, it can be shown that stabilizing and tracking issues of system Eq. (19) will be solved by applying the proposed method in this paper.

4.1- Stabilizing of System States

$\Phi_1(x_1,x_2)$ و $\Phi_2(x_1,x_2)$ are defined as:

$$\Phi_1(x_1,x_2)=-x_1^2-x_2\,,\ \Phi_2(x_1,x_2)=-2x_2^2 \qquad (21)$$
$$k_1=.5 \quad , \quad k_2=3$$

By substituting $\Phi_1(X)$ و $\Phi_2(X)$ to Eq. (4) and (5),:

$$u=-(x_1^2+x_2^2)-k_1(\eta+x_1^2+x_2)-k_2(\eta+2x_2^2) \qquad (22)$$
$$-2x_1\dot{x}_1-(4x_2+1)\dot{x}_2 \quad ; \quad k_1,k_2>0$$

And

$$V_t(x_1,x_2,\eta)=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2+\frac{1}{2}(\eta+x_1^2+x_2)^2 \qquad (23)$$
$$+\frac{1}{2}(\eta+2x_2^2)^2$$

By applying a control signal, *u*, to the system, figures (4), (5) & (6) are resulted.

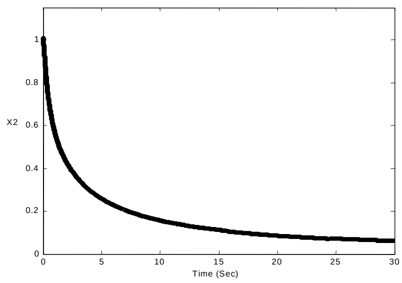

Figure 4- Variations of X_1 After Stabilization

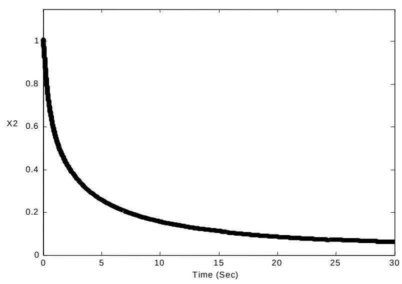

Figure 5- Variations of X_2 After Stabilization

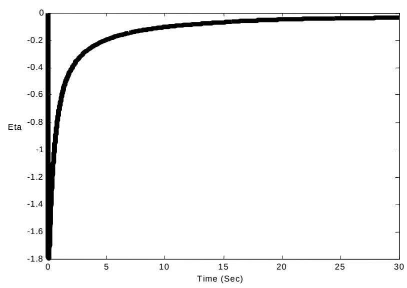

Figure 6- Variations of η After Stabilization

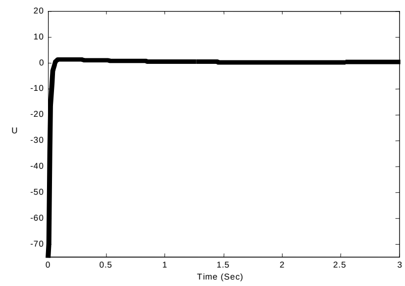

Figure 7- Control Signal *u* for System Stabilization

As it can be seen from figures (4),(5) and (6) for all states the system is stable and reaches to equilibrium point. Figure (7) shows the desired control signal for system stabilization.

4.2 – Tracking Problem

Now it is assumed a step input is applied to the system Eq. (19), where the output of the system is x_1: $\mu=x_1-r(t)$ (24)

By substituting Eq. (24) system Eq. (19) is converted to the following:

$$\begin{cases}\dot{\mu}=(1+\mu)^3-\mu x_2+(1+\mu)\eta \\ \dot{x}_2=x_2^3+x_2\,\eta \\ \dot{\eta}=u\end{cases} \qquad (25)$$

By considering the condition of the mentioned theorem it can be shown that:

$$\Phi_1(\mu,x_2)=\frac{\mu x_2-2(1+\mu)^3}{1+\mu} \quad ,$$
$$\Phi_2(\mu,x_2)=-2x_2^2 \qquad (26)$$
$$k=1 \quad , \quad k_2=2$$

The simulation results are given in figures (8) and (9). As it is seen from these figures the output will follow the system input fairly.

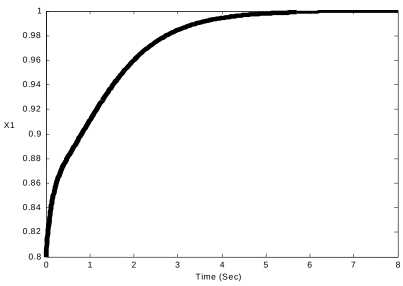

Figure 8- Response to Step Input

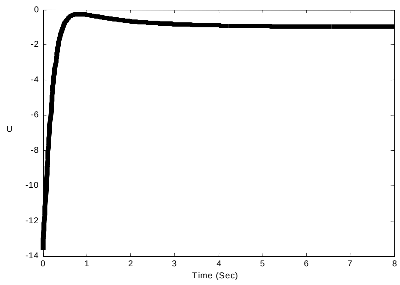

Figure 9- Control Signal *u* for Tracking

5. Conclusions

In this paper a new proposed technique that is generalized back-stepping method is presented. This method can be applied to non-linear control systems properly. By considering the simulation results, it is seen that both stabilization as well as input tracking will be achieved in an acceptable manner. It is mentioned that most of non-linear systems can be modeled using these system equations. The presented model in this paper is a proper methodology to control a wide range of non-linear systems. In fact, generalized back-stepping method can help one to stabilize those unstable states of a system. The results also show the effectiveness of the proposed technique in an acceptable fashion.

References:

[1] O. Harkegard. Flight Control Design Using Backstepping, Division of Automatic Control, 2001.

[2] P.V. Kokotovic. The joy of feedback : Nonlinear and adaptive. IEEE Control Systems Magazine, 12(3) : 7-17 , june 1992.

[3] I. Kanellakopoulos , P. V. Kokotovic , and A. S. Morse. A toolkit for nonlinear feedback design. System & Control Letters, 18(2) : 83-92 , February 1992.

[4] M. Kristic , Ioannis Kanellakopoulos , and Petar Kokotovic. Nonlinear and Adaptive Control Design. John Wiley & Sons , 1995.

[5] R. A. Freeman and Petar V. Kokotovic. Robust Nonlinear Control Design : state-Space and Lyapunov Techniques. Birkhauser, 1996.

[6] R. Sepulchre, M. Jankovic, and P. V. Kokotovic. Interlaced Systems and recursive design for global Stabilization. In Proceedings of the 1997 European Control Conference, 1997.

[7] P.V. Kokotovic. Constructive nonlinear control : Progress in the 90's. In IFAC 1999 Proceeding, Pages 49-77, 1999.

[8] H. K. Khalil. Nonlinear Systems : 2nd Edition.