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On the Computation of Balancing Transformations for a Linear Controllable System

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Abstract

This paper deals with the algorithms for the reduction of the controllability and observability Grammians to block - diagonal forms. The proposed algorithms permit us to avoid the computation of the singular values when solving the problem of reduction of the order of a controllable dynamical system.

Keywords: *Controllability, observability, Grammian, system reduction.*

1. Introduction

Linear system transformations, which diagonalize the controllability and observability Grammians, are effectively used for the reduction of the order of a linear system (see, e.g., [1-3]). For this reason many attempts have been made to improve the procedures related with this kind of transformations [4]. Relying upon the methods of [5,6], an algorithm is presented below for the construction of the transformation which reduces above-mentioned Grammians to block-diagonal forms and, unlike the traditional ways, does not require determination of singular values. The algorithm used in the paper is summarized in the Appendix.

2. Linear Controllable System Transformations [1-3]

We first state the essence of the problem. The motion of a controllable system is described in the state space as:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \epsilon R^n, u \epsilon R^m, y \epsilon R^s$ are respectively the vectors of, internal coordinates, controlling inputs and observable coordinates. Matrices **A, B** and **C** (of corresponding dimensions) do not depend on time. The pairs **(A,**

B), **(C, A)** are assumed to be controllable and observable, and the matrix **A** is stable, i.e. $\mathbf{Re}\lambda(A) < 0$. Controllability and observability Grammians W_r and W_0 satisfy the Lyapunov equations

$$\begin{aligned} AW_r + W_rA^T + BB^T &= 0, \\ A^TW_0 + W_0A + C^TC &= 0, \end{aligned} \tag{2}$$

where T denotes transposition. If we linearly transform system (1) (i.e. passing to new coordinates $\overline{x}$), by taking $x = T\overline{x}$, where $n \times n$ matrix $T \in R^n$ is invertible, the matrices in (2) will have the following forms:

$$\begin{aligned} \overline{A} &= T^{-1}AT, \quad \overline{B} = T^{-1}B, \quad \overline{C} = CT, \\ \overline{W}_r &= T^{-1}W_rT^{-T}, \quad \overline{W}_0 = T^TW_0T, \end{aligned} \tag{3}$$

where index -T denotes inversion and transposition: i.e.$T^{-T} = (T^T)^{-1}$. Expanding W_r into Cholesky factors

$$W_r = WW^T \tag{4}$$

and introducing the orthogonal matrix Q, which diagonalizes W^TW_0W, i.e.

$$\begin{aligned} Q^TW^TW_0WQ &= \Sigma^2 = diag\{\sigma_1^2, \sigma_2^2, ..., \sigma_n^2\}, \\ &\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0 \end{aligned} \tag{5}$$

we can define the family of transformations T_k as follows:

$$T_k = WQ\Sigma^{-k} \tag{6}$$

It is noted in [1,4] that the more interesting cases are the transformations with k=0, k=0.5, k=1. In the first case (k=0) system is normalized with respect to inputs (input-normal), Grammians are $\overline{W} = I_r$ (from now on I is a unit matrix) and $\overline{W}_0 = \Sigma^2$. In the second case of k=0.5, i.e., in the case of balancing transformation (internally balanced), we have $\overline{W}_r = \overline{W}_0 = \Sigma$. And, finally, for k=1 the system is normalized with respect to outputs (output-normal), with $\overline{W}_r = \Sigma^2, \overline{W}_0 = I$. These transformations may be used for the reduction of the order of the system. In fact, let k=0.5, assume that $\sigma_q \gg \sigma_{q+1}$. Since $\overline{W}_r = \overline{W}_0 = \Sigma$, first q $(q < n)$ components of the vector $\overline{x}$ are much better controllable and observable. Dividing matrices of (3) and (5) into blocks in accordance with the division of vector $\overline{x}$ into two components $\overline{x}^T = [\overline{x}_1^T, \overline{x}_2^T]$, $\overline{x}_1 \in R^q$, $\overline{x}_2 \in R^{n-q}$, we obtain:

$$\overline{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \overline{C} = [C_1C_2], \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}. \tag{7}$$

Here block A_{11} is of dimension $q \times q$, and so on. (A_{11}, B_1, C_1) is the most controllable and observable subsystem of the system (A,B,C), and therefore it may be used as a lower order system, approximating (A, B, C). In frequency domain the following inequality is given for the error of this approximation (see, e.g., [2,3]):

$$\| C(jwI - A)^{-1}B - C_1(jwI - A_{11})^{-1}B_1 \|_\infty \leq 2tr\Sigma_2, \tag{8}$$

where $\| x(jw) \|_\infty = sup_{w\epsilon R}\overline{\sigma}\{x(jw)\}$, with $\overline{\sigma}$ being the maximal singular value, and tr is the trace of matrix. Note that the condition $\sigma_q \gg \sigma_{q+1}$, generally speaking, is not the only possible way to demonstrate the leading role of the system (A_{11}, B_1, C_1). It is known (see, e.g., [1,7]) that it is possible to state this property in terms of traces of the matrices Σ_1^2, Σ_2^2. Namely, subsystem (A_{11}, B_1, C_1) is considered to be dominant, if

$$1 \gg \mu^2 = \frac{tr\Sigma_2^2}{tr\Sigma_1^2} \simeq \frac{tr\Sigma_2^2}{trW^T W_0 W},$$

i.e. if the part of the trace of matrix $W^T W_0 W$, corresponding to the trace of matrix Σ_2^2, is small. In [7] there is also another justification of the dominant role of the subsystem.

In view of the aforesaid, there is some freedom in the determination of the dominant subsystem in order to simplify the computation procedures. So, from now on we shall assume that the dominant role of the subsystem (A_{11}, B_1, C_1) is determined by sufficiently small quantity γ such that

$$\sigma_r^2 > \gamma trW^T W_0 W > \sigma_{r+1}^2.$$

Evidently choice of γ determines the value of index r, dimension of the subsystem (A_{11}, B_1, C_1). This determination of the leading role of subsystem (A_{11}, B_1, C_1) enables us to avoid the procedure of the diagonalization of (5) when seeking the transformation, with the help of which the reduction process is simplified. This also permits us to seek transformations which do not diagonalize the Grammians W_r and W_0, but reduce these matrices to block-diagonal forms.

3. Construction of the Transformation, Reducing the Matrix $\mathbf{W^T W_0 W}$ to the Block-diagonal Form

In order to construct the transformations which reduce the Grammians to block-diagonal forms, we begin with the calculation of the transformation τ, reducing the matrix $W^T W_0 W$ to a block diagonal form with the upper diagonal block consisting of the eigenvalues $\sigma_1^2, \sigma_2^2, ..., \sigma_r^2$. For this purpose the algorithm of [5] may be used.

I. Taking the quantity γ, set $\nu = \gamma trW^T W_0 W$.

II. Define $signU_\nu$ as the signum function of the matrix $U_\nu = W^T W_0 W - \nu I$ and the corresponding projectors $sign^+U = 1/2(I + signU_\nu)$, and $sign^-U = 1/2(I - signU_\nu)$ (see, Appendix A). Note, that $tr(sign^+U_\nu) = r$, and $tr(sign^-U_\nu) = n - r$.

III. Construct an $n \times r$ matrix S_+ with the r linearly independent columns of the matrix $sign^+U_\nu$. Similarly, construct an $n \times (n-r)$ matrix S_- by the $n - r$ linearly independent columns of the matrix $sign^-U_\nu$.

VI. We determine the transformation τ, which reduces the matrix U_ν to a block-diagonal form, as:

$$\tau = [S_+, S_-], \quad \tau^{-1}U_\nu\tau = \begin{bmatrix} U_{\nu+} & 0 \\ 0 & U_{\nu-} \end{bmatrix}.$$

Size of the blocks $U_{\nu+}$ and $U_{\nu-}$ are $r \times r$ and $(n-r) \times (n-r)$ respectively, and the eigenvalues are $\sigma_1^2, \sigma_2^2 ..., \sigma_r^2$ and $\sigma_{r+1}, \sigma_{r+2}, ..., \sigma_n$.

4. The Relationship Between Matrices τ and Q

We divide the matrix Q of (5) into the blocks Q_1 and Q_2, with dimensions $n \times r$ and $n \times (n-r)$ respectively, i.e. $Q = [Q_1, Q_2]$. We now show that $S_+ = Q_1 H_1$ and $S_- = Q_2 H_2$, for some invertible matrices H_1, H_2. In fact, from (5) we have

$$U_\nu = Q \begin{bmatrix} \Sigma_1^2 - \nu I & 0 \\ 0 & \Sigma_2^2 - \nu I \end{bmatrix} Q^T.$$

Since the diagonal elements of the matrix $\Sigma_1^2 - \nu I$ are positive, while those of the matrix $\Sigma_2^2 - \nu I$ are negative, we have

$$signU_\nu = Q \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} Q^T.$$

Analogously

$$sign^+ U_\nu = Q_1 Q_1^T, \quad sign^- U_\nu = Q_2 Q_2^T \tag{9}$$

On the other hand, we can permute columns of the matrix $signU_\nu$ (by multiplying it from right by the permutation matrix Π) in such a way that first r columns of the transformed matrix will form the matrix S_+. The remaining columns will be the linear combination of the first r, and consequently, the new matrix will be expressible as the product of matrix S_+ and some other matrix R. Thus,

$$[sign^+ U_\nu]\Pi = [S_+ \quad N] = Q_1 Q_1^T \Pi, \quad N = S_+ R.$$

Since $Q_1^T Q_1 = I$, we have

$$[H_1 \quad H_1 R] = Q_1^T \Pi, \quad H_1 = Q_1^T S_+.$$

Matrix $Q_1^T \Pi$ has full rank, so H_1 is invertible. Multiplying last relation by Q_1 from left and comparing with the previous relation, we find that $S_+ = Q_1 H_1$. By similar reasoning we also find that $S_- = Q_2 H_2$, where H_2 is invertible. Generally speaking, the invertibility of H_1 and H_2 follows from reference [5], because in our case U_ν is diagonalized by Q. Thus

$$\tau = QH, \quad H = diag\{H_1, H_2\}. \tag{10}$$

Using the transformation τ, we construct a matrix that reduces W_0 and W_r to a block-diagonal form. This matrix is

$$\overline{\tau} = W\tau = WQH. \tag{11}$$

In fact, from (3) we have

$$\overline{W}_r = \tau^{-1} W_r \tau^{-T} = diag\{H_1^{-1} H_1^{-T}, H_2^{-1} H_2^{-T}\}, \quad (12)$$
$$\overline{W} = \tau^{T} W_0 \tau = diag\{H_1^{T} \Sigma_1^2 H_1, H_2^{T} \Sigma_2^2 H_2\}. \quad (13)$$

By varying the matrix H in (11), we can obtain various transformations, which provide the blocks of matrices $\overline{W}_r$ and $\overline{W}_0$ in (12), (13) with the corresponding spectral characteristics. For example, we can choose H_1 and H_2 such that $\overline{W}_r = I$, and corresponding eigenvalues of blocks of $\overline{W}_0$ will coincide with the diagonal elements of matrices Σ_1^2, Σ_2^2 (the analog of the transformation (6) for k=0). In the following sections we construct analogs of the transformation (6) for k=0, k=0.5 and k=1.

5. Orthogonalization of the Matrix QH

If matrix H is orthogonal, then transformation (11) is an analog of (6) for k=0. In this case the problem is to construct the transformation (10), in which H as well as QH are orthogonal. For this purpose we can use the orthogonalization procedure of [6,8,9]. Let the singular decomposition [10] of matrices H_1 and H_2 be in the form

$$H_1 = U_1 \Gamma_1 V_1^T, \quad H_2 = U_2 \Gamma_2 V_2^T,$$

where $U_i, V_i (i = 1,2)$ are orthogonal matrices, and

$$\Gamma_1 = diag\{\gamma_{11}, \gamma_{12}, ..., \gamma_{1r}\}, \Gamma_2 = diag\{\gamma_{21}, \gamma_{22}, ..., \gamma_{2q}\},$$
$$q = n - r, \quad \gamma_{11} \geq \gamma_{12} \geq ... \geq \gamma_{1r} > 0, \quad \gamma_{21} \geq \gamma_{22} \geq ... \geq \gamma_{2q} > 0.$$

Hence, (10) may be written as

$$\tau = U \Gamma V^T, U = Q \quad diag\{U_1, U_2\},$$
$$\Gamma = diag\{\Gamma_1, \Gamma_2\}, \quad V = diag\{V_1, V_2\}.$$

Evidently, U and V are orthogonal and Γ is diagonal with positive diagonal elements. If we introduce the sequence

$$2\Phi_{p+1} = \Phi_p + \Phi_p^{-T}, \quad \Phi_0 = \tau, \quad p = 0,1,2.... \quad (14)$$

then, as shown in Appendix B we obtain the result,

$$\Phi_\infty = \lim_{p \to \infty} \Phi_p = UV^T = Q \quad diag\{U_1 V_1^T, U_2 V_2^T\} \quad (15)$$

Thus if take $\tau_0 = W\Phi_\infty$ (i.e. transformation (10), with orthogonal $H_1 = U_1 V_1^T$ and $H_2 = U_2 V_2^T$) as the transforming matrix, then we obtain an analog of the transformation (6) for k=0. In fact, in this case $\overline{W}_r = I$ and

$$\overline{W}_0 = diag\{\overline{W}_{01}, \overline{W}_{02}\}, \quad \overline{W}_{01} = H_1^T \Sigma_1^2 H_1, \quad \overline{W}_{02} = H_2^T \Sigma_2^2 H_2$$

Since H_1 and H_2 are orthogonal, eigenvalues of W_{01} and W_{02} will coincide with diagonal elements of Σ_1^2 and Σ_2^2.

Now we construct an analog of (6) for k=1. Decompose the blocks of the matrix

$$\Phi_\infty^T W^T W_0 W \Phi_\infty = diag\{H_1^T \Sigma_1^2 H_1, H_2^T \Sigma_2^2 H_2\} = LL^T \tag{16}$$

into Cholesky factors. If we take $\tau_1 = W\Phi_\infty L^{-T}$ as the transforming matrix, then we obtain an analog of (6) for k=1. According to (16), singular decomposition of matrix L will be in the form

$$L = diag\{H_1^T \Sigma_1 \Psi_1, H_2^T \Sigma_2 \Psi_2\}, \tag{17}$$

where Ψ_1, Ψ_2 are orthogonal matrices. Thus

$$\begin{aligned} \tau_1^{-1} W_r \tau_1^{-T} &= L^T L = diag\{\Psi_1^T \Sigma_1^2 \Psi_1, \Psi_2^T \Sigma_2^2 \Psi_2\}, \\ \tau_1^T W_0 \tau_1 &= I. \end{aligned}$$

Thus, in view of the spectral characteristics of transformed Grammians, the transformation τ_1 is an analog of the transformation T_1 of (6). Finally if we decompose the blocks of the matrix

$$(\Phi_\infty^T W^T W_0 W \Phi_\infty)^{1/2} = 11^T \tag{18}$$

into Cholesky factors (an algorithm for calculating the square root of a positive definite matrix can be found in Appendix C) and take $\tau_2 = W\Psi_\infty 1^{-T}$ as the transforming matrix, then we obtain the transformation which ensures the equality of eigenvalues of the controllability and the observability Grammians

$$\begin{aligned} \tau_2^{-1} W_r \tau_2^{-T} &= 1^T 1, \\ \tau_2^T W_0 \tau_2 &= 1^T 1. \end{aligned}$$

Taking into account the singular decomposition of the matrix 1 (analogous to (17)) in the form

$$1 = diag\{H_1^T \Sigma_1^{1/2} \Omega_1, H_2^T \Sigma_2^{1/2} \Omega_2\}$$

where Ω_1, Ω_2 are orthogonal matrices, we have

$$1^T 1 = diag\{\Omega_1^T \Sigma_1 \Omega_1, \Omega_2^T \Sigma_2 \Omega_2\},$$

i.e. transformation τ_2 corresponds to transformations (6) when k=0.5. Note that calculating the trace of the lower diagonal block of the matrix $1^T 1$, we can, by use of (8), estimate the error of the approximation corresponding to the chosen γ. Thus, matrices that generate the analogs of the transformation (6) for k=0, k=0.5 and k=1 are

$$\begin{aligned} \tau_0 &= W\Phi_\infty, \\ \tau_2 &= W\Phi_\infty 1^{-T}, \\ \tau_1 &= W\Phi_\infty L^{-T}, \end{aligned}$$

which are determined by means of the matrices in (4), (15), (16) and (18).

6. Computational Procedures

We can further simplify our computations by eliminating the matrix inversion procedures from the operation of finding the signum function of U_ν and the orthogonalization of matrix τ. By [11], if the matrix X satisfies the condition

$$\| I - X^2 \| < 1 \tag{19}$$

where $\| \, . \, \|$ is some matrix norm, then it is possible to dispense with the inversion procedure when computing the signum function. When this condition is satisfied, we have

$$\begin{aligned} signX &= \lim_{p\to\infty} X_p, \quad 2X_{p+1} = X_p(3I - X_p^2), \\ X_0 &= X, \quad p = 0,1,2,... \end{aligned}$$

Since the eigenvalues of matrix U_ν are real, condition (19) will be satisfied if the matrix U_ν is correspondingly scaled. In fact, $signU_\nu = sign(\epsilon U_\nu)$, if $\epsilon > 0$. Thus calculation of $signU_\nu$ may be replaced by the calculation of $sign(\epsilon U_\nu)$ if the factor ϵ is chosen so that $\| I - \epsilon^2 U_\nu^2 \| < 1$. Assuming that the norm in (19) is a spectral norm $\| \, . \, \|_2$ ($\| X \|_2$ is a maximal singular number of the matrix X [10]), we assert that the condition (19) will be satisfied if $\epsilon \| U_\nu \|_2 < \sqrt{2}$. The choice of ϵ may be simplified by use of the M-norm, [10] ($M(X) := n(max_{ij}|X_{ij}|)$, where X_{ij} are elements of the $n \times n$ matrix X). Since $M(X) \geq \| X \|_2$, [10], we can take

$$\epsilon = \frac{1}{M(U_\nu)} < \frac{\sqrt{2}}{\| U_\nu \|_2}.$$

Thus, an algorithm for the calculation of $signU_\nu$ is as follows:

$$\begin{aligned} signU_\nu &= \lim_{p\to\infty} Z_p, \quad 2Z_{p+1} = Z_p(3I - Z_p^2), \\ Z_0 &= \frac{1}{M(U_\nu)} U_\nu, \quad p = 0,1,2,... \end{aligned}$$

A similar algorithm (which also does not require matrix inversion) may also be used in the orthogonalization of the matrix τ. So, introducing the sequence

$$\begin{aligned} &2\Psi_{p+1} = \Psi_p(3I - \Psi_p^T \Psi_p), \quad p = 0,1,2... \\ &\Psi_0 = (1/M(\tau))\tau, \end{aligned} \tag{20}$$

we obtain that the matrix $\Psi_\infty = \lim_{p\to\infty} \Psi_p$ coincides with the matrix Φ_∞ defined by (15) [6]. This orthogonalization scheme can be used in the algorithm for the computation of the square-root of a positive definite matrix, that is to form (18) (another algorithm that includes also the operation of inversion of the matrix is presented in Appendix C). The algorithm has the following steps:

I. Define the Cholesky factor L:

$$LL^T = \Phi_\infty^T W^T W_0 W \Phi_\infty$$

II. Calculate the **M**-norm of L, i.e. $M(L)$.

III. define Ψ_∞ using (20) with

$$\Psi_0 = (1/M(L))L.$$

IV. Find value of the desired root:

$$(\Phi_\infty^T W^T W_0 W \Phi_\infty)^{1/2} = \Psi_\infty L^T$$

7. Conclusion

In this paper an algorithm is given for finding a transformation for a stationary linear controllable system that reduces the controllability and observability Grammians to block diagonal forms. Since the algorithm is related to the calculation of the projectors that are expressed through the signum function of the corresponding matrix it does not require singular decomposition. It is also shown that under certain conditions, matrix inversion can be eliminated from the determination of the matrix signum-function.

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Appendix A: The Matrix signum-function [5,11].

Scalar signum function of a complex variable λ $(Re(\lambda) \neq 0)$ is defined as

$$sign\lambda := \begin{cases} +1 & \text{if } Re\lambda > 0 \\ -1 & \text{if } Re\lambda < 0 \end{cases}$$

This concept can be generalized to the matrix case. Let an $n \times n$ matrix A have no eigenvalues on the imaginary axis, then there exists a matrix M such that

$$J = M^{-1}AM = Block \quad diag\{J_+, J_-\},$$

where the Jordan blocks J_+, J_- have dimensions $n_1 \times n_1$ and $n_2 \times n_2$ $(n_1 + n_2 = n)$ respectively and their eigenvalues lie respectively in the right and left half-planes. Then we define

$$signA = Mdiag\{I_1, -I_2\}M^{-1}. \tag{A.1}$$

where I_1 and I_2 are identity matrices of dimensions n_1 and n_2. Note that

$$(signA)^2 = I \tag{A.2}$$

We also define matrices $sign^+A, sign^-A$ as:

$$sign^+A = \frac{1}{2}[I + signA], \tag{A.3}$$

$$sign^-A = \frac{1}{2}[I - signA]. \tag{A.4}$$

There, (A.3), (A.4) are projectors, since they satisfy the condition [12]

$$(sign^+A)^2 = sign^+A, \quad (sign^-A)^2 = sign^-A,$$

which are easily checked using (A.2). There exist some simple iterations to find the matrix signum function. For example,

$$signA = \lim_{p\to\infty} A_p, \quad 2A_{p+1} = A_p + A_p^{-1}, \quad A_0 = A, \quad p = 0, 1, 2, ... \tag{A.5}$$

Appendix B: An Algorithm for Orthogonalization (polar decomposition) [6,8,9].

Let the singular decomposition of $n \times n$ matrix Φ have the form $\Phi = U\Sigma V^T$, where U, V are orthogonal matrices, Σ is a diagonal matrix with positive diagonal elements. The problem is to find the orthogonal matrices U and V^T. Define the sequence

$$2\Phi_{i+1} = \Phi_i + \Phi_i^{-T}, \quad \Phi_0 = \Phi, \quad i = 0, 1, 2, ... \tag{B.1}$$

The limit of this sequence is UV^T, that is

$$UV^T = \Phi_\infty = \lim_{i\to\infty} \Phi_i. \tag{B.2}$$

Multiplying (B.1) with U^T from left and with V from right we have a sequence of diagonal matrices with positive diagonal elements

$$2\Sigma_{i+1} = \Sigma_i + \Sigma_i^{-T}, \quad \Sigma_0 = \Sigma, \quad i = 0, 1, 2...$$

From (A.1) and (A.5) we have $\lim_{i\to\infty} \Sigma_i = I$, which implies (B.2). There are modifications that increase the speed of convergence. For example the sequence in (B.1) can be replaced with:

$$\begin{aligned} \Phi_{i+1} = &\quad \alpha_i \Phi_i + \beta_i \Phi_i^{-T}, \quad i = 0, 1, 2.... \\ \alpha_i = &\quad [(det\Phi_i)^{1/n} + 1]^{-1}, \beta_i = 1 - \alpha_i. \end{aligned} \tag{B.3}$$

Appendix C: An Algorithm for Finding the Square-root of a Positive Definite Matrix [6].

It is necessary to find $X = A^{1/2}$, that is, the positive definite solution of the equation

$$X^2 - A = 0 \tag{C.1}$$

Since A is positive definite, then it has a Cholesky decomposition

$$A = LL^T.$$

Singular decompositions of the matrices A and L have the following forms

$$A = U\Sigma U^T, \quad L = U\Sigma V^T, \tag{C.2}$$

where U, V are orthogonal matrices and Σ is a diagonal matrix with positive diagonal elements. From (C.2) solution of the problem has the form

$$A^{1/2} = UV^T L^T.$$

Thus, the construction of the root of the equation (C.1) is reduced to the construction of the matrix UV^T, for which one can use, for example, the procedures of (B.1), (B.3) or (20).

Doğrusal Denetlenebilen Sistem için Dengeleme Dönüşümü Hesabı

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Özet

Bu makalede denetlenebilirlik ve gözlenebilirlik Grammian'larının blok köşegenlere indergeme algoritmaları incelenmektedir. Önerilen algoritmalar denetlenebilir bir dinamik sistemin kademe indirgeme probleminin tekil değer hesapları yapılmadan çözülmesini sağlamaktadır.

Anahtar Sözcükler: *Denetlenebilirlik, gözlenebilirlik, Grammian, sistem indirgeme.*