ARBITRARY STABILITY DEGREE FOR UNCERTAIN SYSTEMS WITH INTERNAL DELAY BY LINEAR CONTROL

Eduard Cheres
e-mail: edi@.iec.co.il
Planning Development and Technology Division
The Israel Electric Corporation Ltd.
P.O.B 10 , Haifa, Israel, 31000

Zalman J.Palmor
palmor@tx.technion.ac.il
Faculty of Mechanical Engineering
Technion –Israel Institute of Technology
Technion City, Haifa 32000, Israel.

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ABSTRACT

In this paper we consider the exponential stabilization of control loops with time varying state delay and both matched and unmatched parameter uncertainties. We focus on linear control, and derive new sufficient conditions guaranteeing arbitrary degree of stability. The results could easily be extended to nonlinear controllers of the min-max type.

I. INTRODUCTION

Control loops with time delays are common in chemical processes and in other transport phenomena. The modeling of some of these systems is not a simple task. To accommodate imperfect modeling, uncertain time varying parameters are commonly introduced into the model.

In the chemical and related industries linear controllers are widely used. Therefore there is an interest in the exponential stabilization of uncertain dynamical systems having state delays by linear controllers.

Cheres et al. [1] proposed a class of discontinuous nonlinear controllers which guarantee exponential stability with arbitrary stability degree for systems/control loops with state delays and matched uncertainties (SDMU). But these controllers cannot be directly implemented. Instead they proposed a class of continuous controllers [2], which approximate the discontinuous ones. The synthesis of such a continuous controller that guarantees a desired stability degree requires the knowledge of the bound of the delay variation.

Wu and Mizukami (WM) [3], [4] employed one controller from that class of continuous controllers. In [3] they derived directly sufficient conditions for the exponential stability of those systems but without delays, while in [4] they considered only asymptotic stability of SDMU systems. Sufficient conditions were derived directly for the nonlinear controller. Also they present a class of linear controllers, which retains uniform ultimate boundeness of SDMU systems, when external disturbances are also present. The design of their controllers does not depend on the upper bound of the delay, and this was referred to as a design advantage. But this independence stems from the fact that only stability with zero degree was considered. In practice, however, it is common to design a controller for some positive decay rate. The controllers in [4] are not applicable to such cases.

In this paper we develop sufficient conditions guaranteeing an arbitrary degree of stability for time varying uncertain systems with time varying uncertain state-delays (SDU) and linear controllers. The conditions are derived using the Razumikhin approach [5]. The derived conditions are general, as they are applicable to both matched and unmatched uncertainties, to both systems with state delay and without state delay and for asymptotic as well as exponential stability. In view of the derived conditions, a controller design that is based solely on matched uncertainties [1], [4] may become unstable in the presence of 'slightly' unmatched uncertainties.

The example in [4], which consists of an uncertain time varying system with a time varying state delay, is considered and based upon the conditions derived in the paper a linear controller with a predetermined stability degree is designed and evaluated. This example is also used to compare between the new stability conditions of this paper and those presented in [4], and to demonstrate the influence of the exponential stability on the controller design and the performance of the system.

II. MAIN RESULTS

In this section new sufficient conditions for an arbitrary degree of stability of SDU systems are derived. The uncertain system is represented by the following differential equation:

$$\dot{x} = [A(t) + \Delta A(v,t)]x(t) + \Delta D(w,t)x(t-h) + [B(t) + \Delta B(r,t)]u(t) \quad (1)$$

where: t is the elapsed time, $x(t) \in R^n$ is the current value of the state, u (t) $\in R^n$ is the control function, $(v,w,r) \in \psi$ is the uncertain vector, and $\psi \in R^L$ is a compact set. $A(t), B(t)$ is a continuous uniformly completely controllable pair with appropriate dimensions. The system matrix uncertainties $\Delta A(v,t), \Delta D(w,t), \Delta B(r,t)$ are continuous in all their arguments.

h is any continuous bounded function $0 \le h \le \hat{h}$, where $\hat{h}$ is a known upper delay bound. The uncertain matrices are decomposed as follows:

$$\Delta A(v,t) = B(t)H(v,t) + \Delta\tilde{A}(v,t) \quad (2a)$$

$$\Delta D(v,t) = B(t)F(w,t) + \Delta\tilde{D}(w,t) \quad (2b)$$

$$\Delta B(v,t) = B(t)E(r,t) + \Delta\tilde{B}(r,t) \quad (2c)$$

The stability degree β is now used to define the transformation:

$$z \equiv e^{\beta t} x(t) \quad (3)$$

Differentiating eq. (3) we obtain:

$$\dot{x} = -\beta e^{-\beta t} z + e^{-\beta t} \dot{z} \quad (4)$$

Substituting (3)-(4) into (1) and arranging terms yields:

$$\dot{z}(t) = [A(t) + \beta I + \Delta A(v,t)]z(t) + \Delta D(w,t)e^{\beta h} z(t-h) + [B(t) + \Delta B(r,t)]e^{\beta t} u(t) \quad (5)$$

For clarity purposes we omit the arguments in the sequel. We select (6) as our Lyapunov function:

$$V = z'Pz \quad (6)$$

where the positive definite matrix P is the steady state solution [6] of the following Riccati equation

$$\dot{P} + [A' + \beta I]P + P[A + \beta I] - PBB'P = -I \quad (7)$$

and I is the identity matrix.

The derivative of V along a solution of (5) satisfies:

$$\dot{V} = 2z'P\dot{z} + z'\dot{P}z = -z'z + z'PBB'Pz + 2z'P\Delta Az + 2e^{\beta h} z'P\Delta Dz(t-h) + 2e^{\beta t} z'PBu + 2e^{\beta t} z'P\Delta Bu \quad (8)$$

We synthesize the following linear controller

$$u = -\gamma e^{-\beta t} \alpha \quad (9)$$

with

$$\alpha = B'Pz \quad (10)$$

Substitution of the decomposition (2) and the control law (9)-(10) in (8) yields:

$$\dot{V} = -z'z + \alpha'\alpha + 2\alpha'Hz + 2z'P\Delta\tilde{A}z + 2e^{\beta h}\alpha'Fz(t-h) + 2e^{\beta h} z'P\Delta\tilde{D}z(t-h) - 2\gamma\alpha'\alpha - 2\gamma\alpha'E\alpha - 2\gamma z'P\Delta\tilde{B}\alpha \quad (11)$$

We rewrite (11) as follows

$$\dot{V} = -[z' \quad \alpha'] \begin{bmatrix} aI & -H' \\ -H & 2c(I + \frac{E+E'}{2}) - I \end{bmatrix} \begin{bmatrix} z \\ \alpha \end{bmatrix} - [z'(t-h) \quad \alpha'] \begin{bmatrix} dI & -e^{\beta h}F' \\ -e^{\beta h}F & 2(\gamma - c)\left(I + \frac{E+E'}{2}\right) \end{bmatrix} \begin{bmatrix} z(t-h) \\ \alpha \end{bmatrix} - z'[(1-a) - 2P\Delta\tilde{A} + 2\gamma P\Delta\tilde{B}B'P]z + 2e^{\beta h} z'P\Delta\tilde{D}z(t-h) + dz'(t-h)z(t-h) \quad (12)$$

Where a, c, d are scalars.

We recall that a Rayleigh quotient is bounded by its associated minimal and maximal eigenvalues [7]. Using this property we may write the conditions for positive definiteness [8] of the two matrices in (12) as follows:

$$a > 0 \quad (13)$$

$$2c(1 + \lambda\!\!\!^{-}) - 1 - \frac{\bar{H}^2}{a} > 0 \quad (14)$$

$$d > 0 \quad (15)$$

$$2(\gamma - c)(1 + \lambda\!\!\!^{-}) - \frac{e^{2\beta h}}{d}\bar{F}^2 > 0 \quad (16)$$

and the relevant notations are:

$\lambda[\bullet] \equiv$ eigenvalue of $[\bullet]$;

$\lambda\!\!\!^{-} \equiv \min_r \lambda\left[\frac{E(t,r) + E'(t,r)}{2}\right]$;

$\bar{M}^2(t,r,v,w) \equiv \max_{r,v,w}$ [Rayleigh quotient corresponding to $MM'(t,r,v,w)$]

If $\lambda\!\!\!^{-} > -1$, conditions (14) and (16) are fulfilled by taking

$$2\gamma > \frac{[\frac{\bar{H}^2}{a}+\frac{e^{2\beta h}}{d}\bar{F}^2+1]}{1+\text{ƛ}} \qquad (17)$$

If the negative definite terms in (12) are omitted then

$$\begin{aligned}\dot{V} &< -z'[(1-a)-2P\Delta\tilde{A}+2\gamma P\Delta\tilde{B}B'P]z+ \\ &2e^{\beta h}z'P\Delta\tilde{D}z(t-h)+dz'(t-h)z(t-h)\end{aligned} \qquad (18)$$

Following Razumikhin [5] we assume that the control law does not render the stability of the system. Hence, roughly speaking the function V is increasing along the system trajectories. This can be translated to the following inequality [1], [4]:

$$\|z(t-h)\| < \bar{\lambda}(t)\|z(t)\|, \quad \bar{\lambda}(t) \equiv \sqrt{\frac{\lambda_{\max}[P(t)]}{\lambda_{\min}[P(t)]}} \qquad (19)$$

and $\|\bullet\|$ is the spectral norm.

Inserting (19) into (18) leads to

$$\begin{aligned}\dot{V} &< -z[(1-a)-2P\Delta\tilde{A}+2\gamma P\Delta\tilde{B}B'P]z+ \\ &2e^{\beta\hat{h}}\bar{\lambda}\max_{w}\|P\Delta\tilde{D}\|\|z\|^2+d\bar{\lambda}^2\|z\|^2\end{aligned} \qquad (20)$$

If the right hand side of (20) is non-positive, a contradiction with our assumption that the system is unstable results.

The above discussion is now summarized in the following theorem:

Theorem: Consider system (1), and suppose there exist scalars a, d >0 such that the following conditions are satisfied:

$\text{ƛ}(t)$ > -1 for every t>0. (21)

$$\gamma > \frac{[\frac{\bar{H}^2}{a}+\frac{e^{2\beta\hat{h}}}{d}\bar{F}^2+1]}{2(1+\text{ƛ})} \qquad (22)$$

$$\begin{aligned}&1+\gamma\min_{r}\lambda_{\min}[P\Delta\tilde{B}B'P+PB\Delta\tilde{B}'P]>a+ \\ &\max_{v}\lambda_{\max}[P\Delta\tilde{A}+\Delta\tilde{A}'P]+\bar{\lambda}(2e^{\beta\hat{h}}\max_{w}\|P\Delta\tilde{D}\|+d\bar{\lambda})\end{aligned} \qquad (23)$$

where P is the steady state solution [6] of the Riccati equation (7).

Then the control law:

$$u=-\gamma B'Px(t) \qquad (24)$$

renders system (1) with stability degree β.

Proof: The proof follows from the preceding discussion. Formally the proof may be obtained by using Razumikhin theorems [5], Lemma 1 and Observations 1-2 of [1].

III. EXAMPLE 1: SYNTHESIS OF A ZERO STABILITY DEGREE CONTROLLER

In order to compare between the results of our controller synthesis and the one in [4], the special case of asymptotic stability is discussed in this example. Among other things we demonstrate that, as in [4], the obtained controller is completely independent of the time-varying delay bound.

The time-varying system presented in [4] is discussed:

$$\Delta A = 0 \qquad A=\begin{bmatrix}-1+\frac{2}{1+t} & 1 \\ 0 & 0.5+\frac{1}{1+t}\end{bmatrix}$$

$$\Delta B=\begin{bmatrix}0 \\ 1-0.5\sin(t)\end{bmatrix} \qquad B=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\Delta D=\begin{bmatrix}0 & 0\\ \cos(3t) & \cos(3t)\end{bmatrix}$$

The delay function is given in Fig. 1, $\hat{h}=2$, and the initial conditions of the states are:

$x(t)=[\cos(t) \quad \cos(t)], \quad -\hat{h}\le t\le 0$.

This system has only matched uncertainties, i.e. ,

$\Delta\tilde{A}=\Delta\tilde{D}=\begin{bmatrix}0&0\\0&0\end{bmatrix}, \Delta\tilde{B}=\begin{bmatrix}0\\0\end{bmatrix}$

and the matched parts are given by:

$H=[0 \quad 0]$, $E=1-0.5\sin(t)$,

$F=[\cos(3t) \quad \cos(3t)]$

Thus $\text{ƛ}=0.5$; $\bar{F}^2=2$ and the rest of the uncertain matrices are zero.

If only zero stability degree is required, then inequalities (22)-(23) leads to:

$$\gamma > \frac{2/d+1}{3};\ 0<d<\frac{1}{\bar{\lambda}^2} .$$

Therefore the gain

$$\gamma=\frac{2(\bar{\lambda}^{2}+10^{-3})\ \ +1}{3} \qquad (25)$$

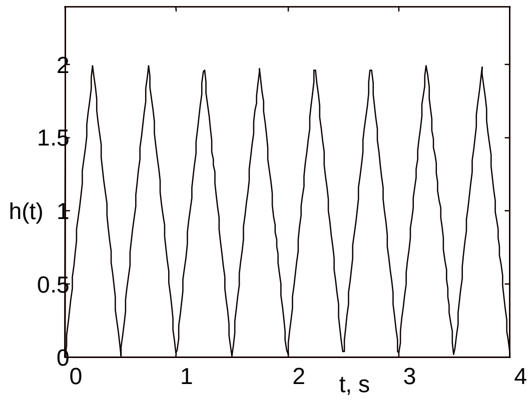

Fig. 1. Time delay history

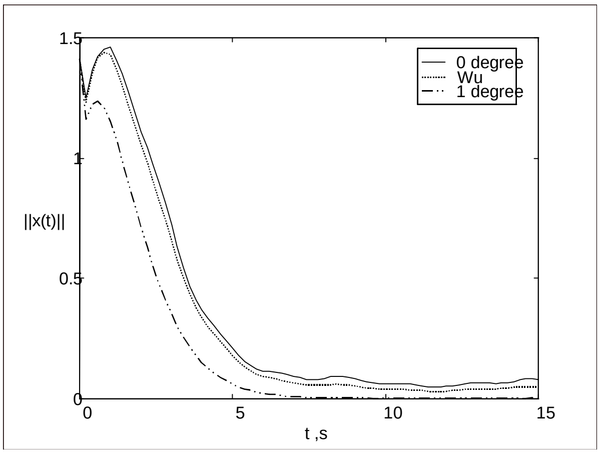

Fig. 2 State norm histories

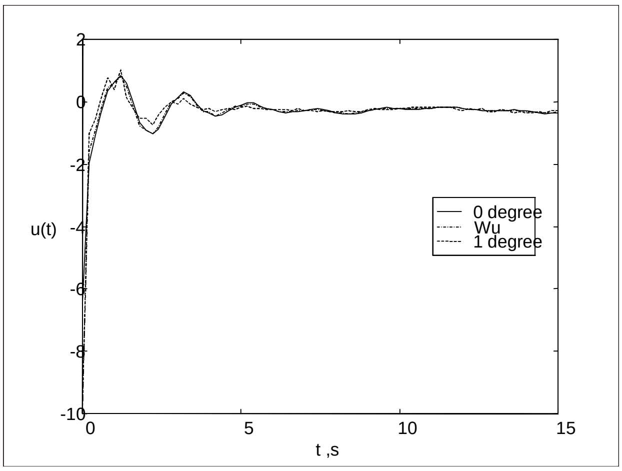

Fig 3. Control histories

stabilizes the system, and clearly γ larger than the one in (25) stabilizes the system too.

Note that WM [4] used the same control but obtained a higher controller gain given by:

$$\gamma = \frac{4.5\bar{\lambda}^2 + 6}{5} \tag{26}$$

If the uncertain matrix ΔB is unmatched, the sign of the expression:
$\min_r \lambda_{\min}[P\Delta\tilde{B}B'P + PB\Delta\tilde{B}'P]$ in condition (23) will dictate which gain to select and whether a stabilizing gain exists. However if no information is given on the uncertainty, the lowest possible gain (25) is preferable in view of inequality (23).

To implement the control law (24) we approximate the steady state solution of the Riccati eq. (7) with $\beta = 0$. It can be approximated, after some time period, by any special solution with a nonnegative definite terminal condition [6]. For comparison we use the same terminal condition as in [4], i.e. the zero matrix. The Riccati eq. is solved backwards in time for 40 s. Since only the steady state portion of the special solution can be used, the simulation of the system is terminated after 15 s.
The state trajectories obtained with our control and the one in [4] are close and are depicted in Fig. 2. Note that the controller is independent of the delay bound. That is, even for extremely large delay values the control will still render the system at zero stability degree.

IV. EXAMPLE 2: SYNTHESIS OF CONTROLLER FOR A STABILITY DEGREE OF 1

We use the system of the previous example, and require a practical decay ratio of the trajectories, i.e. a positive stability degree. For conveniently we require the system to have a stability degree of $\beta = 1$.
This leads to the following inequalities:

$$\gamma > \frac{2e^4/d + 1}{3};\ 0 < d < \frac{1}{\bar{\lambda}^2}\ .$$

Thus a controller gain of at least

$$\gamma = \frac{54.6(\bar{\lambda}^2 + 10^{-3}) + 1}{3} \tag{27}$$

should be used.

The Riccati equation (7) is solved with zero terminal condition and $\beta = 1$. Since the initial condition of the state are arbitrary, the controller gain (27) may become very large at t=0. This is impractical; therefore the controller absolute output value is restricted to be less than 10 (this value was also the maximal controller amplitude in the previous example). The controller outputs are presented in Fig. 3. Note that almost the same output amplitude is required in all cases, but the state trajectories differ significantly. As shown in Fig. 2, the state norm in this case settles down to the 'zero neighborhood' in 10 s, while the settling time for the zero degree of stability controllers is much longer.

V. CONCLUSIONS

In this paper new sufficient conditions for the exponential stability of uncertain time varying systems with state delays and linear control were derived. These conditions generalize all the previous conditions published in the literature. The conditions are applicable to both matched and unmatched uncertainties. It is demonstrated that control gain in the matched cases is bounded from below, whereas in the unmatched cases it may be bound from above depending on the satisfaction of condition (23). The conditions derived here for linear control can be extended to nonlinear control of the min-max type [1] using the same techniques.

REFERENCES

[1] E.Cheres, S. Gutman, and Z.J. Palmor, "Stabilization of uncertain dynamic systems including state delay," *IEEE Trans. Automat. Contr*. , vol. 34, pp. 1199-1203, Nov. 1989.

[2] S.Gutman and Z.J. Palmor, "Properties of min max controller in uncertain dynamical systems," *SIAM J. Contr. Optimiz.*, vol. 20, pp. 850-861, 1982.

[3] H. Wu and K.Mizukami, "Exponential stability of a class of nonlinear dynamical systems with uncertainties," *Systems & Control Letters*, vol. 21,pp. 307-313, 1993.

[4] H. Wu and K. Mizukami,"Linear and nonlinear stabilizing continuous controllers of uncertain dynamical systems including state delay," *IEEE Trans. Automat. Contr*., vol. 41, pp. 116-121, Jan. 1996.

[5] J. Hale, *Theory of Functional Differential Equations.* New York: Springer-Verlag, 1977.

[6] H.Kwakernaak , R. Sivan, *Linear Optimal Control Systems*, John Wiley & Sons, 1972, pp. 230-249.

[7] B. Noble and J.W.Daniel, *Applied Linear Algebra*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2nd ed., 1977, p. 431.

[8] E.Kreindler and A.Jameson, "Conditions for nonnegativeness of partitioned matrices," *IEEE Trans. Automat. Contr*., pp. 147-148, Feb.1972.

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