ON THE BEHAVIOR OF THE INPUT IMPEDANCE OF THE INFINITE LADDER NETWORK

Deniz Pazarcı — e-mail: dpazarci@yeditepe.edu.tr
Ali Ümit Keskin — e-mail: auk@yeditepe.edu.tr
Cevdet Acar — e-mail: acar@yeditepe.edu.tr

Yeditepe University, Faculty of Engineering and Architecture, Department of Electrical and Electronics Engineering, 34755, Kayışdağ, Istanbul, Turkey

Key words: infinite ladder network, input impedance

ABSTRACT

Lines are the interconnections that convey power and information such as speech, music, pictures and data. They can be modeled as infinite ladder structures. Therefore analysis of the infinite ladder networks may be important to describe their features. In this study their input impedance behavior is discussed under specified termination conditions.

I. INTRODUCTION

The input impedance of the ladder network may be important for the calculation of the active and reactive powers delivered to the system. Keeping *n* large enough any line can be analyzed as a cascade connection of *n* identical sections and hence can be modeled as the ladder structure of Figure 1, where Z_a shows the lossy inductance and Z_b shows the lossy capacitance of the line. In the following work, calculations and features of the input impedance of this ladder network are discussed and compared with results reported earlier in [1, 2].

II. MATHEMATICAL ANALYSIS

Consider the finite ladder network of Figure 1 consisting of n identical two-port sections. Each of these sections is composed of impedance Z_a in series and impedance Z_b in parallel. Z_n and Z_0 in Figure 1 are respectively the input and termination (i.e., load) impedances of the ladder network constructed of *n* sections. In the lossy ladder these immitances are $Z_a = r_a + j\omega L$ and $Y_b = g_b + j\omega C = 1/Z_b$, where r_a is resistance and represents the loss of the inductor and g_b is conductance and represents the loss of the capacitance. In the lossless case, r_a and g_b are zero. The input impedance Z_n can be calculated by the following recursive formula:

$$Z_n = Z_a + \frac{Z_b Z_{n-1}}{Z_b + Z_{n-1}} \qquad (1)$$

First of all, let $Z_0 \rightarrow \infty$, corresponding to the unterminated case. In this case the input impedance or equivalently the driving-point impedance, Z_n of Figure 1 or Figure 2 can be written in the Cauer I form [3] as:

$$Z_n = Z_a + \cfrac{1}{Y_b + \cfrac{1}{Z_a + \cfrac{1}{Y_b + \ldots}}} \qquad (2)$$

In the lossless case Z_n of Equation (2) is expressed as:

$$Z_n = \frac{V_n}{I_n} = j\omega L + \cfrac{1}{j\omega C + \cfrac{1}{j\omega L + \cfrac{1}{j\omega C + \ldots}}} \qquad (3)$$

where V_n and I_n are the phasor voltage and the phasor current, respectively. Note that, the lossless case Z_n of Equation (3) is in the form of algebraic sums of imaginary functions. As a result Z_n is also an imaginary function, i.e., it has no real part as expected for a lossless LC network. In other words, unless a lossless LC ladder network is terminated by an impedance with a nonzero real part, its input impedance can not have a nonzero real part. This complies with the conservation of energy.

Second, consider Figure 3 where *n*=1 and Z_0 is the termination impedance. It can be proven that Z_1 equals Z_0 only if Z_0 satisfies the following quadratic equation:

$$Z_0^{\,2} - Z_a Z_0 - Z_a Z_b = 0 \qquad (4)$$

from where Z_0 is found as:

$$Z_0 = \frac{Z_a + \sqrt{Z_a^{\,2} + 4Z_aZ_b}}{2} \tag{5}$$

If the circuit in Figure (3) is terminated by an impedance given by Equation (5), then this same impedance is going to be seen from the input. In the lossless case, Z_0 is:

$$Z_0 = \frac{V_0}{I_0} = \frac{j\omega L + \sqrt{-\omega^2L^2 + 4\frac{L}{C}}}{2} \tag{6}$$

Next, consider again the case of n cascaded identical sections as in Figure 1. Let us find the condition for Z_{k+1} to be equal to Z_k. In this case,

$$Z_{k+1} - Z_k = Z_a + \frac{Z_kZ_b}{Z_k + Z_b} - Z_k = 0 \tag{7}$$

which results in the quadratic expression of Equation (4), whose solution is the same as Equation (5). Therefore, for $Z_{k+1} = Z_k$ to be true, the $(k+1)^{\text{th}}$ stage must be terminated by $Z_k = Z_0$ of Equation (5). In other words, for the input impedance of the ladder network to remain unchanged after one more section is added, the special termination of Equation (5) is required. If this is satisfied, then the input impedances of all stages will equal the termination impedance, Z_0. i.e.,

$$Z_k = \frac{V_k}{I_k} = Z_0 = \frac{Z_a + \sqrt{Z_a^{\,2} + 4Z_aZ_b}}{2} \tag{8}$$

for k=1, 2, ... , n. Note that Equation (8) is independent of the number of stages cascaded. Furthermore, if we cascade an infinite number of sections, the input impedance remains unchanged:

$$Z_{in} = \lim_{n\to\infty}\frac{V_n}{I_n} = Z_0 = \frac{Z_a + \sqrt{Z_a^{\,2} + 4Z_aZ_b}}{2} \tag{9}$$

In the lossless case Z_{in} is:

$$Z_{in} = \lim_{n\to\infty}\frac{V_n}{I_n} = Z_0 = \frac{j\omega L + \sqrt{-\omega^2L^2 + 4\frac{L}{C}}}{2} \tag{10}$$

This input impedance has a nonzero real part when $\omega < \omega_c = 2/\sqrt{LC}$, which is given by:

$$\operatorname{Re}\{Z_{in}\} = \frac{\sqrt{-\omega^2L^2 + 4\frac{L}{C}}}{2} \tag{11}$$

This is not a surprise because the ladder network is terminated by Z_0 of Equation (6), which also has a nonzero real part for the same frequency range.

III. CONCLUSION

For the case of an unterminated lossless infinite LC ladder, Feynman et al. [1] claimed that $Z_{k+1} = Z_k$ and Equation (9) is valid. According to their study the input impedance of such an unterminated ladder has a resistive part for $\omega < \omega_c = 2/\sqrt{LC}$. Another paper [2] calls this the paradoxical behavior of the infinite LC ladder. However, in the unterminated ($Z_0 \to \infty$) case, Z_1 becomes $Z_1 = j\omega L + 1/j\omega C$, which does not satisfy the second order equation, resulting in $Z_{k+1} \neq Z_k$. i.e., the input impedance is not Z_0 of Equation (6) any more. In addition, the Cauer I form clearly shows that the input resistance can not have a resistive part for all ω in a lossless LC ladder, as mentioned before. In conclusion, our work proves [1] to be erroneous.

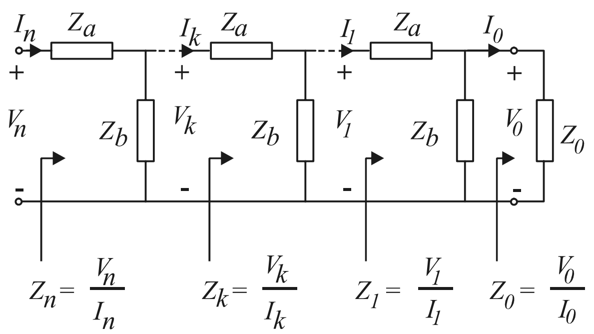


Figure 1. Two-port ladder network with n identical sections terminated with Z_0.

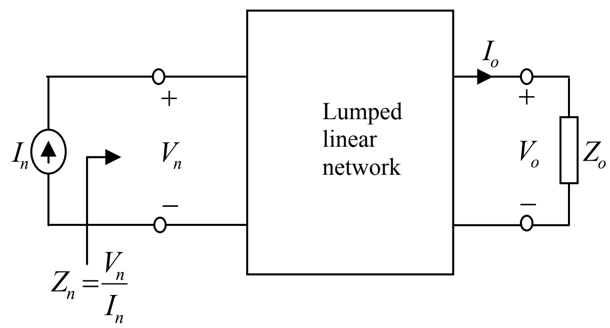
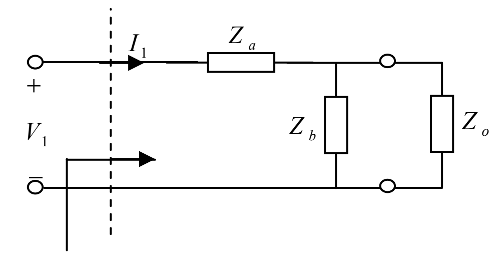


Figure 2. Driving-point impedance, Z_n .



$$Z_1 = \frac{V_1}{I_1} = Z_0$$

Figure 3. Single two-port section terminated with Z_0.

REFERENCES

1. R.Feynman, R.B.Leighon, M.L. Sands, The Feynman lectures on physics, commemorative issue. (Originally published in 1964 by Addison Wesley, Reading, MA), 1989, Vol.2, Chapter 22.
2. S. J. van Enk, “Paradoxical behavior of an infinite ladder network of inductors and capacitors”, American J. Physics, 68, 854-856, (2000).
3. G. C. Temes/J.W. LaPatra, “Introduction to Circuit Synthesis and Design”, McGraw Hill, 1977.