A SENSITIVITY MEASURE FOR AN ELECTRONIC, PROPORTIONAL-INTEGRAL-DERIVATIVE- (PID) CONTROLLER AND CALCULATING OPTIMUM PARAMETER TOLERANCES

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Abstract
The PID controllers are one of the most important control elements used in process control industry. In this study, first a method is proposed to determine the optimum parameter tolerances by the use of the sensitivities of an electronic PID controller's transfer function. These tolerances keep the relative error of the output of the PID controller due to parameter variations, in tolerance region. Then a sensitivity measure is defined to be used to improve the sensitivity performance of the PID controller and to compare various PID controllers with different set of design parameter values which realize the same transfer function.

1. Introduction

Over the last 15 years, control theory has developed several techniques for designing linear time-invariant control systems that are optimum and robust. It is shown that optimum and robust controllers, designed by these techniques can produce extremely fragile controllers, in the sense that vanishingly small deviations of the coefficients, due to the environmental effects, of the designed controller destabilize the closed-loop system. The fragility also shows up usually as extremely small gain and/or phase margins of the closed-loop system [1].

It is obvious that it would be unwise to place a controller that is fragile with respect to deviations of its coefficients due to the environmental effects in an actual control system without further precautions and analysis since deviations of controller coefficients cause the deviation of the output voltage of the controller.

Although reducing the deviations of the output voltage of a controller due to the environmental effects is very important in designing and in using such elements, this problem has not been examined and solved so far besides an electronic, integral-type controller [2].

In this study a method is proposed to overcome this problem by use of the sensitivity concept. For this purpose, using transfer function of an electronic PID controller and the parameter sensitivities, first an upper bound is given for the deviation of the output voltage. Then the optimum parameter tolerances which satisfy this upper bound are presented. And a sensitivity measure is defined for the PID controller.

2. Evaluation of output deviation

Let us consider the electronic, proportional-integral-derivative controller shown in Fig.1.

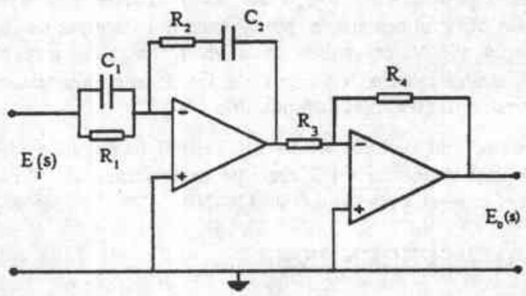

Figure 1. An Electronic, Proportional-Integral-Derivative(PID) Controller

The transfer function between input and the output voltages of the PID controller can be obtained in s-domain as [3],

$$\frac{E_0(s)}{E_i(s)} = T(s) = \frac{R_4(R_1C_1 + R_2C_2)}{R_1R_3C_2} + \frac{R_4}{R_1R_3C_2s} + \frac{R_2R_4C_1}{R_3}s \qquad (1)$$

where $e_0(t)$: output voltage ,
$e_i(t)$: input voltage ,

Using T(s) of (1), the Laplace transform of the output voltage of the PID controller can be found as

$$E_0(s) = T(s)E_i(s) \,. \qquad (2)$$

The relative deviation of the output voltage in terms of parameter sensitivities can be written as follows [4,5]:

$$\frac{\Delta E_0(s)}{E_0(s)} = \frac{\Delta T(s)}{T(s)} = \sum_{i=1}^{6} S_{x_i}^{T}(s)\frac{\Delta x_i}{x_i} \quad , \qquad (3)$$

where

$$x_i \in \{R_1, R_2, R_3, R_4, C_1, C_2\} \quad ,$$
$$\frac{\Delta x_i}{x_i} \in \left\{\frac{\Delta R_1}{R_1}, \frac{\Delta R_2}{R_2}, \frac{\Delta R_3}{R_3}, \frac{\Delta R_4}{R_4}, \frac{\Delta C_1}{C_1}, \frac{\Delta C_2}{C_2}\right\} , \qquad (3a)$$

and $S_{x_i}^{T}(s)$ is the normalized sensitivity of the transfer function, $T(s)$ with respect to ith parameter x_i:

$$S_{x_i}^{T}(s) = \frac{x_i}{T}\frac{\partial T}{\partial x_i} . \qquad (4)$$

The sensitivity, $S_{x_i}^{T}(s)$ can also be written in terms of gain and phase sensitivities as follows [6], for s = jω:

$$S_{x_i}^{T}(\omega) = S_{x_i}^{|T|}(\omega) + jS_{x_i}^{\beta}(\omega) , \qquad (5)$$

where $S_{x_i}^{|T|}$ denotes normalized sensitivity of the gain function:

$$S_{x_i}^{|T|}(\omega) = \frac{x_i}{|T|}\frac{\partial |T|}{\partial x_i} = \mathrm{Re} S_{x_i}^{T}(\omega) , \qquad (6)$$

and $S_{x_i}^{\beta}$ denotes semi-normalized sensitivity of the phase function:

$$S_{x_i}^{\beta}(\omega) = x_i \frac{\partial \beta}{\partial x_i} = \mathrm{Im} S_{x_i}^{T}(\omega) . \qquad (7)$$

Using (1) and (4), the following sensitivities can be obtained:

$$S_{R_1}^{T} = -1/D_1(s) \qquad (8a)$$

$$S_{R_2}^{T} = R_2C_2s/D_2(s) \qquad (8b)$$

$$S_{R_3}^{T} = -1 \qquad (8c)$$

$$S_{R_4}^{T} = 1 \qquad (8d)$$

$$S_{C_1}^{T} = R_1C_1s/D_1(s) \qquad (8e)$$

$$S_{C_2}^{T} = -1/D_2(s) \qquad (8f)$$

where $D_1(s)$, $D_2(s)$ are defined as follows:

$$D_1(s) = 1 + sR_1C_1 , \qquad (9a)$$

$$D_2(s) = 1 + sR_2C_2 . \qquad (9b)$$

Setting s = jω in equations (8), the normalized sensitivities of the gain function, $S_{x_i}^{|T|}$ and the semi-normalized sensitivities of the phase function, $S_{x_i}^{\beta}$, can be obtained respectively, for i=1,..,6 as follows:

$$S_{R_1}^{|T|} = -1/N_1(\omega), \qquad (10a)$$

$$S_{R_2}^{|T|} = R_2^2C_2^2\omega^2/N_2(\omega), \qquad (10b)$$

$$S_{R_3}^{|T|} = -1, \qquad (10c)$$

$$S_{R_4}^{|T|} = 1, \qquad (10d)$$

$$S_{C_1}^{|T|} = R_1^2C_1^2\omega^2/N_1(\omega) , \qquad (10e)$$

$$S_{C_2}^{|T|} = -1/N_2(\omega) , \qquad (10f)$$

$$S_{R_1}^{\beta} = R_1C_1\omega/N_1(\omega), \qquad (10g)$$

$$S_{R_2}^{\beta} = -R_2C_3\omega /N_2(\omega), \qquad (10h)$$

$$S_{R_3}^{\beta} = 0, \qquad (10j)$$

$$S_{R_4}^{\beta} = 0, \qquad (10k)$$

$$S_{C_1}^{\beta} = R_1C_1\omega/N_1(\omega), \qquad (10l)$$

$$S_{C_2}^{\beta} = R_2C_2\omega/N_2(\omega) . \qquad (10m)$$

where $N_1(\omega)$ and $N_2(\omega)$ are denominator polynomials and are defined as follows::

$$N_1(\omega) = D_1(j\omega)\, D_1(-j\omega) , \qquad (11a)$$

$$N_2(\omega) = D_2(j\omega)\, D_2(-j\omega) . \qquad (11b)$$

The overall relative deviation in output voltage due to parameter variations can be obtained from (3) and (5) as follows:

$$\frac{\Delta E_0(j\omega)}{E_0(j\omega)} = \sum_{i=1}^{6}\left[S_{x_i}^{|T|}(\omega) + jS_{x_i}^{\beta}(\omega)\right]\frac{\Delta x_i}{x_i} . \qquad (12)$$

Substitution of equations (10) into (12) yields

$$\frac{\Delta E_0(j\omega)}{E_0(j\omega)} = \left\{ \begin{array}{l} [(-1+jR_1C_1\omega)/N_1(\omega)]\frac{\Delta R_1}{R_1} \\ +[(R_2^2C_2^2\omega^2+jR_2C_2\omega)/N_2(\omega)]\frac{\Delta R_2}{R_2} \\ +[-1]\frac{\Delta R_3}{R_3}+[1]\frac{\Delta R_4}{R_4} \\ +[(R_1^2C_1^2\omega^2+jR_1C_1\omega)/N_1(\omega)]\frac{\Delta C_1}{C_1} \\ +[(-1+jR_2C_2\omega)/N_2(\omega)]\frac{\Delta C_2}{C_2} \end{array} \right\} \quad (13)$$

With this formula, designer can evaluate the relative deviation of the output of the PID controller, once parameter variations are known.

3. Calculation of optimum parameter tolerances

In practice it is very difficult to predict the parameter variations but their upper bounds named as their tolerances. Hence it is quite hard to estimate the output deviation. Therefore in the following, first an upper bound is given for $\Delta E_0/E_0$. And then using this upper bound the optimum parameter tolerances are calculated.

By the use of the triangular inequality [7,8], (3) and (5) together, the upper bound for overall relative deviation in output voltage can be expressed as follows:

$$\left|\frac{\Delta E_0(j\omega)}{E_0(j\omega)}\right| \le \sum_{i=1}^{6}\left[\sqrt{\left(S_{x_i}^{|T|}(\omega)\right)^2+\left(S_{x_i}^{\theta}(\omega)\right)^2}\right]t_{x_i}\ . \quad (14)$$

Substituting equations (10) into (14), the following bound is obtained for overall relative deviation at any ω in a specified frequency band $\omega\in[\omega_1,\omega_2]$:

$$\left|\frac{\Delta E_0(j\omega)}{E_0(j\omega)}\right| \le \left\{ \begin{array}{l} \left[(1+R_1^2C_1^2\omega^2)^{1/2}/N_1(\omega)\right]t_{R_1} \\ +\left[(R_2^4C_2^4\omega^4+R_2^2C_2^2\omega^2)^{1/2}/N_2(\omega)\right]t_{R_2} \\ +t_{R_3}+t_{R_4} \\ +\left[(R_1^4C_1^4\omega^4+R_1^2C_1^2\omega^2)^{1/2}/N_1(\omega)\right]t_{C_1} \\ +\left[(1+R_2^2C_2^2\omega^2)^{1/2}/N_2(\omega)\right]t_{C_2} \end{array} \right\} \le t_0\ , \quad (15)$$

where t_{x_i} and t_0 are respectively tolerances of ith parameter and variation of the output voltage E_0 defined as follows:

$$\max\left|\frac{\Delta x_i}{x_i}\right| = t_{x_i}\ . \quad (14)$$

$$t_0 = \max\left[\sum_{i=1}^{6}\left|S_{x_i}^{T}(j\omega)\right| t_{x_i}\right],\quad \omega\in[\omega_1,\omega_2]\ . \quad (15)$$

Equation (15) can be used in calculation the optimum parameter tolerances keeping $|\Delta E_0/E_0|\le t_0$ in a specified frequency band as follows:

In ideal PID controller, it is desirable that the parameter variations equally contribute to the output variation [9]. Considering this fact we define the optimum parameter tolerances as

$$t_{x_i} = t_0/\left|S_{x_i}^{T}(\omega_c)\right|,\quad i=1,..6 \quad (16)$$

where ω_c is critical angular frequency at which $\sum_{i=1}^{6}\left|S_{x_i}^{T}(\omega)\right|$ takes its maximum value.

4. Sensitivity Measure Of The System

According to Blostein's definition [10] a sensitivity measure, M_{PID} can be calculated as

$$M_{PID} = \sum_{i=1}^{6}\left|S_{x_i}^{T}\right|\ . \quad (17)$$

This sensitivity measure, M_{PID} can be used not only to improve the sensitivity performance of an electronic, proportional-derivative controller but also allows to compare various proportional-derivative controllers with the same input-output relation.

5. Example

As an application of the above study, in the folowing the optimum tolerances will be calculated according to the proposed formula with the following parameter values in Fig.1.

$$R_1 = R_2 = 153.85K\Omega\ ,\quad R_3 = 10K\Omega\ ,$$
$$R_4 = 197.1K\Omega\ ,\quad C_1 = C_2 = 10\mu F\ . \quad (19)$$

Assuming that designer wants $|\Delta E_0/E_0|$ to be less than or equal to 0.01 then the parameter tolerances are chosen as follows:

$$t_{R_1} = t_{R_2} = t_{C_1} = t_{C_2} = \%1.4\ ; \quad (20)$$
$$t_{R_3} = t_{R_4} = \%1\ .$$

And the sensitivity measure can be calculated as,

$$M_{PID} = 4.822\ . \quad (21)$$

6. Conclusion

In this study, using the sensitivities of the transfer function of a PID controller, a method is proposed to determine the optimum parameter tolerances. These tolerances keep the relative error of the output of the PID controller due to parameter variations in its tolerance region. Furthermore a sensitivity measure is defined to be used to improve the sensitivity performance of the PID controller and to compare various PID controllers with different set of design parameter values which realize the same transfer function.

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