Estimating Autoregressive Moving Average Model Orders of Non-Gaussian Processes

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Abstract

In statistical signal processing, parametric modeling of non-Gaussian processes experiencing noise interference is a very important research area. The autoregressive moving average (ARMA) model is the most general and important tool of modeling system. This paper develops an algorithm for the selection of the proper ARMA model orders. The proposed technique is based on forming a third order cumulant matrix from the observed data sequence. The observed sequence is modeled as the output of an ARMA system that is excited by an unobservable input, and is corrupted by zero-mean Gaussian additive noise of unknown variance. Examples are given to demonstrate the performance of the proposed algorithm.

1. Introduction

In statistical signal processing, parametric modeling of non-Gaussian processes experiencing noise interference is a very important research area. The autoregressive moving average (ARMA) model is the most general and important tool of modeling system. The problem of ARMA model order selection has been of considerable interest and has been applied in many diverse fields such as economics, engineering, signal modeling, and biomedical signal processing [1]. In most practical cases, the model order is not known. This vital and crucial step is ignored, chosen rather arbitrarily, or assumed to be available in many of the commonly employed ARMA modeling algorithms. For example, in spectrum analysis and modeling, the problem of model order selection is of most importance [2]. That is because the accuracy of the frequency estimates depends on the estimated order of the prediction filter [3].

Several methods have been reported in the literature. For example, Akaik [4, 5] proposed an information criterion (AIC) and the final prediction error (FPE). The minimum description length (MDL) was proposed by Rissanen [6] and Schwarz [7]. Parzen [8] proposed the criterion autoregressive transfer function (CAT) method. One method by Liang *et. al.* [9] is shown to yield a level of performance for a general ARMA model order estimation never before achieved. This method is derived from the minimum description length (MDL) principle [6, 7]. It is based on the minimum eigenvalue (MEV) of a family of covariance matrices computed from the observed data. The MEV method does not require prior estimation of the model parameters that means fewer computations than the other MDL-based algorithms. Liang *et. al.* showed that the MDL did not work well at low signal-to-noise ratio (SNR) and is computationally expensive. This is due to the prediction error used in computing MDL that is directly affected by the accuracy of the parameter estimates. Al-Smadi and Wilkes [10] extended the MEV method in [9] (EMEV) to the third order cumulants sequence.

Higher order statistics (HOS), or cumulants, have received attention in signal processing (see [11], [12] and reference therein). Research in HOS has been in existence for almost four decades. Several papers have been published over the past 30 years dealing with the applications of HOS and especially that of the bispectrum. Examples are geophysics, biomedicine, telecommunications, speech processing, and economic time series [12]. The growth of research in digital signal processing with HOS has been explosive during the past 15 years. That is because Cumulants are generally nonsymmetrical functions of their arguments. Hence, cumulants carry phase information about the ARMA transfer functions. Therefore, cumulants are capable of determining the order of ARMA models that contain all-pass (i.e., phase only) factors inherent in ARMA models. Also, cumulants are capable of identifying non-minimum phase systems and reconstructing non-minimum phase signals if the signals are non-Gaussian. In addition, cumulants of order greater than 2 of a Gaussian process vanish. Hence, cumulants provide a measure of non-Gaussianity.

In this paper, we present a new approach to the problem of ARMA model order estimation by utilizing theoretical ideas. The proposed algorithm is based on the minimum eigenvalue of a data covariance matrix derived from the observed data sequence using third-order cumulants. The observed sequence is modeled as the output of an ARMA system that is excited by an unobservable input, and is corrupted by zero-mean Gaussian additive noise. A comparison will be presented between the new algorithm and the EMEV method [10] for different SNRs on the output signal.

2. Problem Formulation

Let a signal $s(t)$ is an ARMA(p,q) process if it is stationary and satisfies the following equation:

$$s(t) = -\sum_{i=1}^{p} a_i s(t-i) + \sum_{i=0}^{q} b_i w(t-i) \tag{1}$$

where $s(t)$ is the output signal and $w(t)$ is the excitation sequence. The excitation signal $w(t)$ is assumed to be zero-mean, non-Gaussian, independent and identically distributed (i.i.d.) process. The parameters $a_0,\ldots, a_p$ are the AR parameters; the number of AR parameters is the order p. The parameters $b_0,\ldots,b_q$ are the MA parameters; q is the MA order. Equation (1) can be written symbolically in the more compact form

$$A_p(z^{-1})s(t) = B_q(z^{-1})w(t) \tag{2}$$

where z^{-1} is the unit delay operator $[\, z^{-k} s(t) = s(t-k)\,]$

$$A_p(z^{-1}) = 1 + \sum_{k=1}^{p} a_k z^{-k}$$

$$B_q(z^{-1}) = 1 + \sum_{i=1}^{q} b_i z^{-i} \quad (3)$$

The roots of the polynomial $A_p(z^{-1})$ are denoted the poles of the ARMA process. The roots of $B_q(z^{-1})$ are the zeros. Processes are called stationary if all poles are within the unit circle, and they are invertible if all zeros are within the unit circle [13]. We model the noisy output as

$$y(t) = s(t) + v(t) \quad (4)$$

where $v(t)$ is additive Gaussian noise.

Now, the system in Equation (1) can be written in matrix format as

$$D_{pq}\, \underline{\theta} = \underline{v} \quad (5)$$

where D_{pq} is a composite data matrix such that

$$D_{pq} = [D_p\ D_q] \quad (6)$$

$\underline{\theta}$ is the coefficients vector,

$$\underline{\theta} = [1\ a_1\ \dots\ a_p\ {-1}\ {-b_1}\ \dots\ {-b_q}]^T \quad (7)$$

and $\underline{v}$ represents the modeling error. The data covariance matrix is obtained as

$$R_{pq} = [D_{pq}]^T D_{pq} \quad (8)$$

Liang's *et. al.* MEV method [9] is based on the MDL [6, 7] and leads to the criterion

$$J_{MEV}(p,q) = \lambda_{\min}\, (N^{1/N})^{(p+q)} \quad (9)$$

where $\lambda_{\min}$ is the minimum eigenvalue of R_{pq}, p is the number of AR parameters, q is the number of MA parameters, N is the length of the observed noisy sequence.

The MEV criterion calculates a table of $J_{MEV}(p,q)$ for all values of p and q. The table is organized so that p increases from left to right while q increases from top to bottom down the table. The search method utilizes row- and column-ratio tables. The tables are formed by dividing each row/column of the $J_{MEV}(p,q)$ by the previous row/column. An estimate of the AR order, p, is set equal to the column number that contains the minimum value of column ratio table. Similarly, the MA order, q, is set equal to the number of the row having the minimum value of the row ratio table.

Recently, Al-Smadi and Wilkes [10] proposed an extended MEV criterion to (EMEV) using third-order cumulants. The extension was made by multiplying both sides of Equation (1) by $s(t+m)s(t+n)$ and taking the expectation which results in

$$C_{sss}(m,n) + a_1 C_{sss}(m+1,n+1) + \dots + a_p C_{sss}(m+p,n+p) = C_{wss}(m,n) + b_1 C_{wss}(m+1,n+1) + \dots + b_q C_{wss}(m+q,n+q) \quad (10)$$

The system in (10) can be represented in a matrix form analogous to (5), that is

$$C_{pq}^{(3)}\ \underline{\theta} = \underline{v}^{(3)} \quad (11)$$

where $\underline{v}^{(3)}$ represents modeling error in the cumulant domain, and

$$C_{pq}^{(3)} = [\, C_{sss}^{(3)} \quad C_{wss}^{(3)}\,] \quad (12)$$

with $C_{sss}^{(3)}$ containing the cumulants of the observed output sequence and $C_{wss}^{(3)}$ containing the cross-cumulants of the input and output sequences. Hence, the data covariance matrix of third order cumulants is

$$\Phi = [\, C_{pq}^{(3)}\,]^T C_{pq}^{(3)} \quad (13)$$

Notice that Φ is symmetric and positive semidefinite. The EMEV criterion becomes

$$J(p,q) = \psi_{\min}\, (N^{1/N})^{(p+q)} \quad (14)$$

where $\psi_{\min}$ is the minimum eigenvalue of the third order cumulant covariance matrix Φ. In the ideal case, if p and q are chosen such that $p \geq p_t$ and $q \geq q_t$ (where p_t and q_t are the true orders), then $\underline{v}^{(3)} = 0$ and from (11) and (13) it follows that $\psi_{\min}$ will be zero [10]. That is,

$$J = \begin{bmatrix} J(0,0) & J(1,0) & \cdots & J(p_t,0) & \cdots & \cdots & J(p_{\max},0) \\ J(0,1) & J(1,1) & \vdots & J(p_t,1) & \vdots & \vdots & J(p_{\max},1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ J(0,q_t) & J(1,q_t) & \vdots & J(p_t,q_t) & \cdots & \cdots & J(p_{\max},q_t) \\ \vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ J(0,q_{\max}) & J(1,q_{\max}) & \cdots & J(p_t,q_{\max}) & 0 & \cdots & 0 \end{bmatrix}$$

(15)

for $p = 0,1,\dots, p_{max}$ and $q = 0,1,.., q_{max}$. Hence, the J matrix forms an infinite flat plane of zeros in the pq-plane. Therefore, the correct model order lies in the corner of this flat plane. To look for the corner where $J(p,q)$ drops sharply, the EMEV method utilizes the row/column ratio tables as in the MEV method.

We will now investigate a new method to locate the corner that can be justified. The method is based on theoretical viewpoints and is derived from the third order covariance matrix, J, in Equation (15). Notice that if p (or q) is selected less than p_t (or q_t), then the modeling error will be significant. That is because the model does not have enough parameters to fit the available signal very well. Hence, the J will be significantly large. In theory, the corner (p_t,q_t) can be detected by transforming the cost function into two vectors; namely, row vector and column vector.

The column vector will have an estimate of the order of the AR part while the row vector will have an estimate of the order of the MA part. The proposed method proceeds as follows.

2.1. Autoregressive Order

The AR order is determined by transforming the J matrix into a row vector. Starting at the left column in (15), the entries of each column are multiplied together to give one value. That is, the multiplication of the elements of each column will give one value for each column. Since we have p_{max} columns, we will have p_{max} values in this row vector. The entries of this vector will be nonzero up to column p_t. That is, multiplication of the elements of any column vector in the region $p>p_t$ will be theoretically zero. Hence, the row vector will contain a number of p_t nonzero entries that is the AR order. Therefore, the number of the entry at which the elements of this vector change from nonzero to zero is considered an estimate for the order of the AR part.

In practice, the case is not ideal. Therefore, the error will not be zero but will be small. Hence, most of the entries in the J matrix will not be zeros in the region $p>p_t$. Thus, in order to locate the cell that determines the proper AR order, the following procedure was developed. Each entry in the row vector is divided by the previous value. The location of the maximum drop between two successive entries in the row vector is used to determine the correct orders for the AR parts.

2.2. Moving Average Order

The MA order is determined by transforming the J matrix into a column vector. Starting at the top row in (16), the entries of each row are multiplied together to give one value. That is, the multiplication of the elements will give one value for each row. Since we have q_{max} rows, we will have q_{max} values in this column vector. The entries of this vector will be nonzero up to row q_t. Hence, multiplication of the elements of any row vector in the region $q>q_t$ will be theoretically zero. Thus, the column vector will contain a number of q_t nonzero entries. Therefore, the number of the entry at which the elements of this vector changes from nonzero to zero is considered an estimate for the order of the MA part.

In practice, the case is not ideal. Therefore, the error will not be zero but will be small. Hence, most of the entries in the J matrix will not be zeros in the region $q>q_t$. Thus, in order to locate the cell that determines the proper MA order, the following procedure was developed. Each entry in the column vector is divided by the previous value. The location of the maximum drop between two successive entries in the column vector is used to determine the correct orders for the MA parts.

3. Simulation Examples

The proposed ARMA model order selection from only the observed noisy output data has been tested on a number of simulated examples. A number of experiments were performed with different seeds for each experiment. In these experiments, the proposed method has been compared with the EMEV method at different signal to noise ratio (SNR) on the output sequence. The computations were performed in MATLAB. A finite length of N=2000 points was considered in each experiment. The driving input sequence is not observed. However, it is needed for computing third order cross-cumulants to construct the matrix $C_{wss}^{(3)}$. Therefore, the technique in [10] was used to estimate the input sequence.

Example 1: The time series to be considered is given by [14]

$$s(t) - 0.8\, s(t-1) + 0.65 s(t-2) = w(t) + w(t-2) \qquad (16)$$

This model has two poles and two zeros, ARMA(2,2). The poles are located at 0.4 ± j0.7. The zeros are located at ±j. The observed time series is $y(t) = s(t) + v(t)$. The excitation sequence $w(t)$ consists of zero-mean and i.i.d. exponential distribution. The noisy output was generated with Gaussian measurement additive noise at different SNRs. The ARMA model order was then estimated by performing 100 independent simulations for both techniques. Each simulation trial has noise with different seeds. The results of both techniques are displayed in Table 1.

Example 2: The time series to be considered is given by [15]

$$s(t) - s(t-1) + 0.5 s(t-2) = w(t) - 2w(t-1) + 2w(t-2) \qquad (17)$$

This model has two poles and two zeros. The poles are located at 0.5 ± j0.5, and the zeros at 1 ± j. Note that this is a rather difficult example, since the model contains an inherent *all-pass factor*. The noisy output was generated as in the previous examples. Then the ARMA model orders were estimated using the EMEV and the proposed methods. However, the driving input sequence was assumed to be observed. A comparison of the EMEV and the proposed method is displayed in Table 2.

Table 1. Model order estimation results for Example 1

	Number of correct estimates	
SNR(dB)	EMEV method	Proposed method
-10	0	2
-8	4	6
-6	12	14
-4	14	16
-2	32	36
0	54	64
2	74	84
4	86	90
6	100	100

Table 2. Model order estimation results for Example 2

	Number of correct estimates	
SNR(dB)	EMEV method	Proposed method
2	8	8
3	28	28
4	32	32
5	60	64
6	84	86
7	96	100
8	100	100

4. Conclusions

A new approach for selecting the model order for ARMA models has been presented. The method presented is based on the minimum eigenvalue of a data covariance matrix derived from the observed data sequence using third-order cumulants.

The proposed algorithm searches for the cell that locates the orders (*p* and *q*) in the tabulation of the minimum eigenvalue of the third order covariance matrix *J*. Numerical examples were given that illustrated the high accuracy of the results that can be obtained with this approach.

5. References

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