A PERFORMANCE ANALYSIS FOR DMUSIC

Ersen Yılmaz Erdoğan Dilaveroğlu

e-mail: ersen@uludag.edu.tr *e-mail: dilaver@uludag.edu.tr*

Uludağ University, Faculty of Engineering and Architecture, Department of Electronic Engineering, 16059, Görükle, Bursa, Turkey

Key words: Frequency estimation ,damped sinusoids, DMUSIC, first order analysis

ABSTRACT

Analytical performance study is provided for DMUSIC algorithm. Theoretical variance expression is derived in matrix form for estimating the frequency parameters. It is demonstrated that theoretical variance expression is closely approximate the exact variance of DMUSIC.

I. INTRODUCTION

The problem of estimating the parameters of damped sinusoidal signals in the presence of additive noise has received significant attention in signal processing literature. The difficulty of the problem stems from the fact that the damped sinusoidal signals is nonstationary and correlation matrix can not be found [1]. Several algorithms have been proposed to solve this problem.

Damped MUSIC (DMUSIC) is one of these algorithms and it was proposed in [2]. It is called DMUSIC because it looks like MUSIC algorithm. But, there are several crucial differences between MUSIC and DMUSIC algorithms in that MUSIC is for parameter estimation of undamped sinusoidal signals. A numerical performance study of DMUSIC and an application for 2D NMR signals can be found in [2]-[3].

In this paper, First Order Analysis (FOA) is provided for DMUSIC and theoretical variance expression is presented in matrix form for estimating the frequency parameters. Then, performance analysis of DMUSIC is investigated analytically and compared with the numerical results. CRB is used as a yardstick in the performance analysis.

It is known that under weak conditions, the variance of any unbiased estimate is always bounded below by CRB [4]-[5]. Accordingly, the CRB frequently is used as a benchmark for assessing the performance of practical estimators.

II. MODEL DESCRIPTION

Data model consists of p complex damped sinusoids in the presence of additive noise is given below:

$$y(k) = \sum_{i=1}^{p} \alpha_i e^{-\beta_i k} e^{j(\omega_i k + \varphi_i)} + e(k)\,,\ k = 0,\ldots,N-1 \quad (1)$$

where α_i is the amplitude, φ_i is the phase, β_i is the damping factor, $\omega_i \in (0,2\pi)$ is the frequency of the i. complex damped sinusoid, $(i = 1,2,\ldots,p)$, $e(k)$ represents the additive noise and N is total number of data samples.

III. DMUSIC [2,3]

DMUSIC is based on the singular value decomposition of the prediction matrix and use directly rank-deficiency and the Hankel properties of the prediction matrix in the estimation. Frequency and damping factor are estimated simultaneously by 2-D search in DMUSIC.

Data model (1) is rearranged as below:

$$y(k) = \sum_{i=1}^{p} \alpha_i e^{s_i k} + e(k)\,,\ k = 0,1,\ldots,N-1 \quad (2)$$

where $s_i = -\beta_i + j\omega_i$, $\beta_i \in R^+$ and $\omega_i \in (0,2\pi)$. DMUSIC algorithm can be derived only if data matrix is set up in a structural way. To derive the DMUSIC algorithm prediction matrix $\mathbf{A}$ is set up as below:

$$\mathbf{A} = \begin{bmatrix} y(0) & y(1) & . & y(L-1) \\ y(1) & y(2) & . & y(L) \\ . & . & . & . \\ y(L-1) & y(L) & . & y(2L-2) \end{bmatrix} \quad (3)$$

where $\min(N-L, L) \geq p$.

From (3) $\mathbf{A}$ can be written as:

$$\mathbf{A} = \sum_{i=1}^{p} c_i \underline{r}(s_i)\underline{r}^T(s_i) + \mathbf{N} = \mathbf{SCS}^T + \mathbf{N} \tag{4}$$

where $\underline{r}(s_i)$ and $\mathbf{S}$ are signal vector and signal matrix that are defined as respectively:

$$\underline{r}(s_i) = \begin{bmatrix} 1 \\ e^{s_i} \\ . \\ e^{(L-1)s_i} \end{bmatrix} \text{ and } \mathbf{S} = \begin{bmatrix} \underline{r}(s_1) & \underline{r}(s_2) & \dots & \underline{r}(s_p) \end{bmatrix} \tag{5}$$

$\mathbf{C}$ is pxp diagonal matrix with $diag(\mathbf{C}) = (c_1, c_2, .., c_p)$ and $\mathbf{N} = \left[e(i+j) \right]_{i,j=0}^{L-1}$ is the noise matrix.

If s_i 's are distinct, then $\underline{r}(s_i)$ for $i = 1,2,...p$ are linear independent and hence $\mathbf{S}$ is full column rank. Since rank of $\mathbf{C}$ is p, the rank of $\mathbf{A}$ is equal to p if there is no measurement noise. Now assume that there is no noise. By means of singular value decompositon $\mathbf{A}$ can be decomposed into the product of three matrices.

$$\mathbf{A} = \mathbf{UDV}^H \tag{6}$$

where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, $\mathbf{D}$ is diagonal matrix with the elements below:

$$diag(\mathbf{D}) = (\rho_1, \rho_2, ..., \rho_p, 0, ..., 0)\ , \quad \rho_1 \ge \rho_2 \ge ... \ge \rho_p \tag{7}$$

From (6)

$$\mathbf{AV} = \mathbf{UD} \tag{8}$$

From (6) and (7) we have the following orthogonality relations:

$$\mathbf{AV_n} = 0 \text{ or } \mathbf{A}\underline{v}_k = 0 \text{ for } k = p+1, ..., L \tag{9}$$

where $\mathbf{V_n} = \left[\underline{v}_{p+1}, ..., \underline{v}_L \right]$.

From (4)

$$\mathbf{SCS}^T \underline{v}_k = 0 \text{ for } k = p+1, ..., L \tag{10}$$

Since both $\mathbf{S}$ and $\mathbf{C}$ are full rank the following equalites can be written:

$$\mathbf{S}^T \underline{v}_k = 0 \text{ for } k = p+1, ..., L \tag{11}$$

$$\underline{r}(s_i)^T \underline{v}_k = 0\ , \quad k = p+1, ..., L \text{ and } n = 1, ..., p \tag{12}$$

Hence, $\mathbf{V_n}^T \underline{r}(s) = 0$ only when $s = s_1, s_2, ..., s_p$. Therefore s_i can be obtained by finding s which makes $\left\| \mathbf{V_n}^T \underline{r}(s) \right\| = 0$. When noise exist, orthogonality relations (9) no longer hold. In this case, signal vectors that most closely orthogonal to noise subspaces are searched. Hence, s_i or $\{\omega_i, \beta_i\}$ can be obtained by finding the peak of the following spectrum:

$$f(\omega, \beta) = f(s) = \frac{1}{\tilde{\underline{r}}^H(s)\mathbf{V_n}^+\mathbf{V_n}^T \tilde{\underline{r}}(s)} \tag{13}$$

where $\tilde{\underline{r}} = \frac{r}{\|\underline{r}\|}$, ‘ $+$ ’ is complex conjugate and ‘ T ’ is transpose.

IV. FIRST ORDER ANALYSIS (FOA)

As $\{\hat{\omega}, \hat{\beta}\}$ is a minimum point of $f(\omega, \beta)$, we must have

$$\left. \frac{\partial f(\omega, \beta)}{\partial \omega} \right|_{\substack{\omega = \hat{\omega} \\ \beta = \hat{\beta}}} = f_\omega(\hat{\omega}, \hat{\beta}) = 0 \tag{14}$$

$$\left. \frac{\partial f(\omega, \beta)}{\partial \beta} \right|_{\substack{\omega = \hat{\omega} \\ \beta = \hat{\beta}}} = f_\beta(\hat{\omega}, \hat{\beta}) = 0 \tag{15}$$

After the first order Taylor series expansion of $f_\omega(\hat{\omega}, \hat{\beta})$ and $f_\beta(\hat{\omega}, \hat{\beta})$ at $\{\omega_\times, \beta_\times\}$, $\omega_\times \in \{\omega_i\}_{i=1}^{p}, \beta_\times \in \{\beta_i\}_{i=1}^{p}$, $(\hat{\omega} - \omega_\times)$ and $(\hat{\beta} - \beta_\times)$ estimation errors can be expressed as below:

$$\begin{bmatrix} (\hat{\omega} - \omega_\times) \\ (\hat{\beta} - \beta_\times) \end{bmatrix} = \mathbf{A}^{-1} \underline{h} \tag{16}$$

From (16), variance of DMUSIC for estimating the frequency can be obtained as follow:

$$\mathrm{var}(\hat{\omega}) = E[(\hat{\omega} - \omega_\times)^2] = E[(\mathbf{A}^{11} h_{11} + \mathbf{A}^{12} h_{21})^2] \tag{17}$$

where $\mathbf{A}^{ij} = (\mathbf{A}^{-1})_{ij}$, $(i, j = 1,2)$ and E is expectation operator.

$\mathbf{A}$ and $\underline{h}$ are given below :

$$\mathbf{A} = \begin{bmatrix} \left. \frac{\partial f_\omega(\hat{\omega}, \hat{\beta})}{\partial \omega} \right|_{\substack{\hat{\omega} = \omega_\times \\ \hat{\beta} = \beta_\times}} & \left. \frac{\partial f_\omega(\hat{\omega}, \hat{\beta})}{\partial \beta} \right|_{\substack{\hat{\omega} = \omega_\times \\ \hat{\beta} = \beta_\times}} \\ \left. \frac{\partial f_\beta(\hat{\omega}, \hat{\beta})}{\partial \omega} \right|_{\substack{\hat{\omega} = \omega_\times \\ \hat{\beta} = \beta_\times}} & \left. \frac{\partial f_\beta(\hat{\omega}, \hat{\beta})}{\partial \beta} \right|_{\substack{\hat{\omega} = \omega_\times \\ \hat{\beta} = \beta_\times}} \end{bmatrix} \tag{18}$$

$$\underline{h} = \left[-f_{\omega}(\omega_{\times},\beta_{\times}) \quad -f_{\beta}(\omega_{\times},\beta_{\times})\right]^T \tag{19}$$

Elements of $\mathbf{A}$ and $\underline{h}$ are given below:

$$A_{11} = 2\,\mathrm{Re}\begin{bmatrix} \underline{d_{\omega}}^*(\omega_{\times},\beta_{\times})\hat{\mathbf{V}}_{\mathbf{n}}{}^{+}\hat{\mathbf{V}}_{\mathbf{n}}{}^{T}\underline{d_{\omega}}(\omega_{\times},\beta_{\times}) \\ +\underline{\tilde{r}}^*(\omega_{\times},\beta_{\times})\hat{\mathbf{V}}_{\mathbf{n}}{}^{+}\hat{\mathbf{V}}_{\mathbf{n}}{}^{T}\underline{d'_{\omega}}(\omega_{\times},\beta_{\times}) \end{bmatrix} \tag{20}$$

$$A_{12} = A_{21} = 2\,\mathrm{Re}\begin{bmatrix} \underline{d_{\beta}}^*(\omega_{\times},\beta_{\times})\hat{\mathbf{V}}_{\mathbf{n}}{}^{+}\hat{\mathbf{V}}_{\mathbf{n}}{}^{T}\underline{d_{\omega}}(\omega_{\times},\beta_{\times}) \\ +\underline{\tilde{r}}^*(\omega_{\times},\beta_{\times})\hat{\mathbf{V}}_{\mathbf{n}}{}^{+}\hat{\mathbf{V}}_{\mathbf{n}}{}^{T}\underline{d_{\omega,\beta}}(\omega_{\times},\beta_{\times}) \end{bmatrix} \tag{21}$$

$$A_{22} = 2\,\mathrm{Re}\begin{bmatrix} \underline{d_{\beta}}^*(\omega_{\times},\beta_{\times})\hat{\mathbf{V}}_{\mathbf{n}}{}^{+}\hat{\mathbf{V}}_{\mathbf{n}}{}^{T}\underline{d_{\beta}}(\omega_{\times},\beta_{\times}) \\ +\underline{\tilde{r}}^*(\omega_{\times},\beta_{\times})\hat{\mathbf{V}}_{\mathbf{n}}{}^{+}\hat{\mathbf{V}}_{\mathbf{n}}{}^{T}\underline{d'_{\beta}}(\omega_{\times},\beta_{\times}) \end{bmatrix} \tag{22}$$

$$\underline{h}_{11} = -2\,\mathrm{Re}\left[\underline{\tilde{r}}^*(\omega_{\times},\beta_{\times})\hat{\mathbf{V}}_{\mathbf{n}}{}^{+}\hat{\mathbf{V}}_{\mathbf{n}}{}^{T}\underline{d_{\omega}}(\omega_{\times},\beta_{\times})\right] \tag{23}$$

$$\underline{h}_{21} = -2\,\mathrm{Re}\left[\underline{\tilde{r}}^*(\omega_{\times},\beta_{\times})\hat{\mathbf{V}}_{\mathbf{n}}{}^{+}\hat{\mathbf{V}}_{\mathbf{n}}{}^{T}\underline{d_{\beta}}(\omega_{\times},\beta_{\times})\right] \tag{24}$$

Where ' $*$ ' is complex conjugate transpose.

$\underline{\tilde{r}}(\omega_{\times},\beta_{\times})$, $\underline{d_{\omega}}(\omega_{\times},\beta_{\times})$, $\underline{d'_{\omega}}(\omega_{\times},\beta_{\times})$, $\underline{d_{\beta}}(\omega_{\times},\beta_{\times})$, $\underline{d'_{\beta}}(\omega_{\times},\beta_{\times})$, $\underline{d_{\omega,\beta}}(\omega_{\times},\beta_{\times})$ vectors in (20)-(24) are given below:

$$\underline{r}(\omega_{\times},\beta_{\times}) = \left[1 \quad e^{-\beta_{\times}+j\omega_{\times}} \quad \ldots \quad e^{(-\beta_{\times}+j\omega_{\times})(L-1)}\right]^T \tag{25}$$

$$\underline{\tilde{r}}(\omega_{\times},\beta_{\times}) = \frac{\underline{r}(\omega_{\times},\beta_{\times})}{\left\|\underline{r}(\omega_{\times},\beta_{\times})\right\|} \tag{26}$$

$$\underline{\tilde{r}}^*(\omega_{\times},\beta_{\times}) = (\underline{\tilde{r}}(\omega_{\times},\beta_{\times}))^* \tag{27}$$

$$\underline{d_{\omega}}(\omega_{\times},\beta_{\times}) = \frac{\partial\underline{\tilde{r}}(\omega_{\times},\beta_{\times})}{\partial\omega} \tag{28}$$

$$\underline{d_{\omega}}^*(\omega_{\times},\beta_{\times}) = (\underline{d_{\omega}}(\omega_{\times},\beta_{\times}))^* \tag{29}$$

$$\underline{d'_{\omega}}(\omega_{\times},\beta_{\times}) = \frac{\partial\underline{d_{\omega}}(\omega_{\times},\beta_{\times})}{\partial\omega} \tag{30}$$

$$\underline{d_{\beta}}(\omega_{\times},\beta_{\times}) = \frac{\partial\underline{\tilde{r}}(\omega_{\times},\beta_{\times})}{\partial\beta} \tag{31}$$

$$\underline{d_{\beta}}^*(\omega_{\times},\beta_{\times}) = (\underline{d_{\beta}}(\omega_{\times},\beta_{\times}))^* \tag{32}$$

$$\underline{d'_{\beta}}(\omega_{\times},\beta_{\times}) = \frac{\partial\underline{d_{\beta}}(\omega_{\times},\beta_{\times})}{\partial\beta} \tag{33}$$

$$\underline{d_{\omega,\beta}}(\omega_{\times},\beta_{\times}) = \frac{\partial^2\underline{d_{\omega}}(\omega_{\times},\beta_{\times})}{\partial\omega\partial\beta} \tag{34}$$

Vector derivatives in (25)-(34) are element-wise derivative taken with the respect to scalar as in (35):

$$\frac{\partial \underline{r}}{\partial\theta} = \left[\frac{\partial r_1}{\partial\theta} \quad \frac{\partial r_2}{\partial\theta} \quad \ldots. \quad \frac{\partial r_{L-1}}{\partial\theta}\right]^T \tag{35}$$

V. PERFORMANCE ANALYSIS

Performance of the DMUSIC is measured by the theoretical variance expression (17). Theoretical variance is compared with the exact variance that is calculated numerically and the CRB. In numerical example we assume that the data model (1) consists of two complex damped sinusoids in the presence of complex AWGN with σ^2 variance. Non-matrix CRB expressions for two complex damped sinusoids are presented in [6].

Example: Consider the case of $N = 100$, $\alpha_1 = \alpha_2 = 1$, $\varphi_1 = \varphi_2 = 0$, $\omega_1 = 1$ and $\omega_2 = \omega_1 + \delta\omega$ with four different choices of frequency difference ($\delta\omega$) and damping factors:

(a) $\delta\omega = 0.5\Omega$, $N\beta_1 = 0.01$, $N\beta_2 = 0.01$
(b) $\delta\omega = 0.5\Omega$, $N\beta_1 = 1$, $N\beta_2 = 0.01$
(c) $\delta\omega = 1.5\Omega$, $N\beta_1 = 0.01$, $N\beta_2 = 0.01$
(d) $\delta\omega = 1.5\Omega$, $N\beta_1 = 1$, $N\beta_2 = 0.01$

Where $\Omega = \frac{2\pi}{N}$, Fouirer resolution limit.

Results are for 500 Monte Carlo trials. The SNR used in example is defined as $\frac{\alpha_i^2}{\sigma^2}$, $i = 1,2$.

Figures are for frequency estimation of the first damped sinusoid. In figures the solid, dashed and dotted curves depict the CRB, exact variance of DMUSIC and theoretical variance of DMUSIC respectively.

VI. CONCLUSION

We have provided FOA for DMUSIC and derived theoretical variance expression for estimating the frequency parameters. It is demonstrated by a numerical example that theoretical variance expression closely approximate the exact variance of DMUSIC starting from the acceptable SNR values for which the frequency difference is below the Fouirer resolution limit and from the low SNR values for which the frequency difference is upper the Fourier resolution limit.

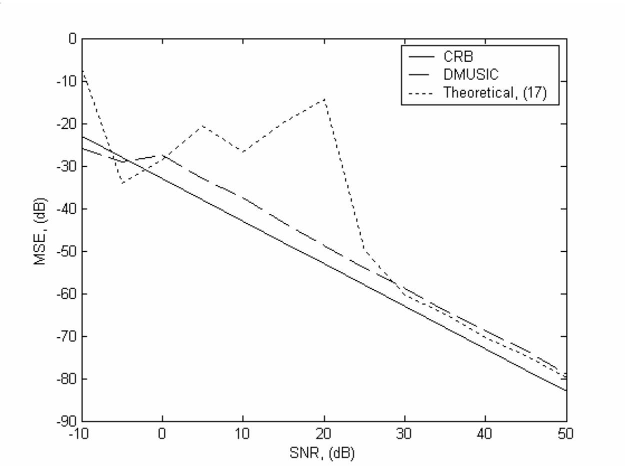

Figure 1. Theoretical and exact variances of DMUSIC and the CRB for frequency estimation of first complex damped sinusoid when $\delta\omega = 0.5\Omega$, $N\beta_1 = 0.01$, $N\beta_2 = 0.01$.

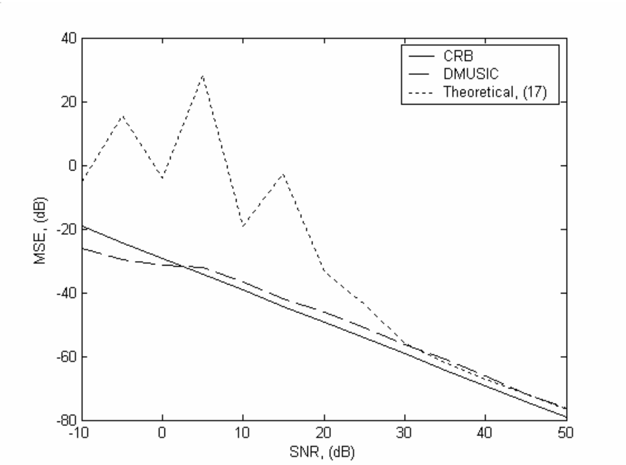

Figure 2. Theoretical and exact variances of DMUSIC and the CRB for frequency estimation of first complex damped sinusoid when $\delta\omega = 0.5\Omega$, $N\beta_1 = 1$, $N\beta_2 = 0.01$.

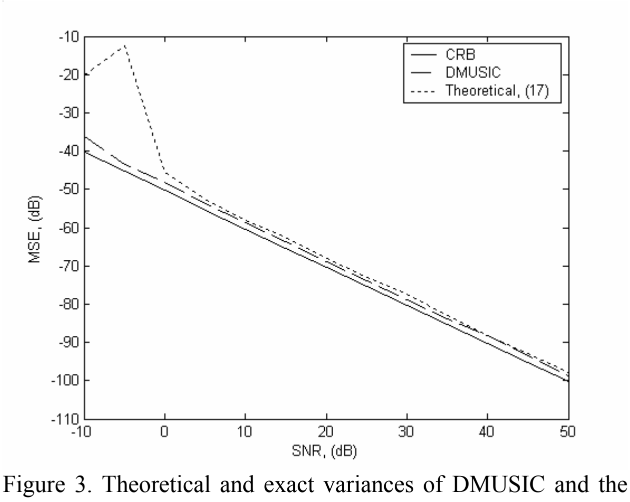

Figure 3. Theoretical and exact variances of DMUSIC and the CRB for frequency estimation of first complex damped sinusoid when $\delta\omega = 1.5\Omega$, $N\beta_1 = 0.01$, $N\beta_2 = 0.01$.

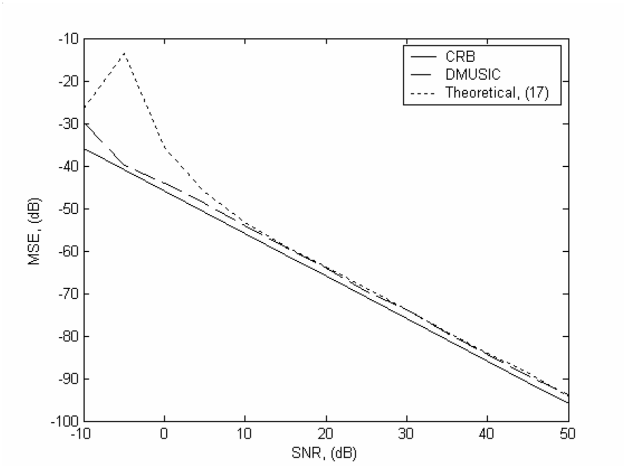

Figure 4. Theoretical and exact variances of DMUSIC and the CRB for frequency estimation of first complex damped sinusoid when $\delta\omega = 1.5\Omega$, $N\beta_1 = 1$, $N\beta_2 = 0.01$.

REFERENCES

1. L. Ye, K. J. R. Liu, J. Razavilar, A Parameter Estimation Scheme for Damped Sinusoidal Signals Based on Low-Rank Hankel Approximation, IEEE Transactions on Signal Processing, Vol. 45, No.2, pp. 481-486, 1997.
2. L. Ye, K. J. R. Liu, J. Razavilar, Improved Parameter Estimation Scheme for Damped Sinusoidal Signals, Technical Report of SRC of University of Maryland, T.R. 95.11, 1995.
3. L. Ye, K. J. R. Liu, J. Razavilar, DMUSIC algorithm for 2D NMR signals, *Engineering in Medicine and Biology Society, IEEE 17th Annual Conference*, Vol. 1, pp.477 – 478, 1995.
4. H. Cramér, Mathematical Methods of Statistics. Princeton, NJ:Princeton Univ. Press, 1946.
5. H. L. Van Trees, Detection, Estimation and Modulation Theory, Part I. New York: Wiley, 1968.
6. E. Yılmaz, ve E. Dilaveroğlu, Non-matrix Cramér-Rao Bound Expressions for Two Complex Damped Sinusoids, Proceedings of 13th IEEE Signal Processing and Communication Applications Conference, 2005.