PDEs-Based Gaussian Noise Removal from Color Images

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Abstract

This study presents a Gaussian noise removal method using the improved trace-based approach for color images. The presented method employs partial differential equations (PDEs)-based approach including both smoothing (regularization) term and fidelity (data) term in the energy functional. So, the structure of the input image is well preserved during noise removal processes. Also, we estimate the standard deviation of Gaussian noise in the wavelet domain. In addition, due to the fact that the type of Gaussian noise is easily perceived via a selected block in a flat (homogeneous) region of the noisy input image, the presented method is considered as a semi-automated noise removal approach. The validity of the presented method is depicted via experimental results.

1. Introduction

Image noise is an undesirable effect that appears in random variations in intensities of a gray-level image or each channel component of a multi-valued image. Non-ideal sensor elements, any adverse environmental situation such as high temperature, and transmission and compression processes can cause a corruption as a kind of noise in the input image. Noises are usually evaluated in two main categories: additive and multiplicative. The most common additive noise types, which are frequently encountered in digital images, are Gaussian noise (sensor noise) and impulsive noise (transmission noise). A low-pass filter (LPF) can be used for eliminating Gaussian noise, and a classical median filter can be used for removal of impulsive noise (salt and pepper noise). However, structure and texture information of the restored image may be inevitably blurred during noise removal processes. For this reason, there have been many approaches for image denoising in order to preserve both structure and texture information of the input image in the literature.

In terms of PDEs, in the literature, many approaches have been developed to eliminate especially additive noises in the input image. Tikhonov [1] developed a linear and isotropic method based on Laplace equation, which is the most basic approach for noise reduction in gray-level images. Due to the fact that this method behaves as a LPF [2], it blurs edges of the input image. Therefore, Perona and Malik [3] included a weighting function to Laplace equation in order to preserve well edge information in the input image. As a result of this, diffusion is prevented by this non-linear anisotropic method in the edge regions of the input image. However, this method may sometimes make the noise more evident instead of reduced noises, which is called inverse diffusion. Weickert [4] presented an anisotropic method that uses diffusion tensors based on the local features extracted from eigenvectors and eigenvalues related to the eigenvectors. Gilboa et.al. [5] proposed a nonlinear diffusion method for image denoising, which preserves both structure and texture information of gray-level images. Another study suggested by Gilboa and Osher [6] uses non-local linear diffusion operators for image regularization. Tschumperle and Deriche [7] presented a trace-based anisotropic method for color image denoising. Due to the fact that this method reveals rounding effects in the sharp corners of the input image, Tschumperle improved it based on the line integral convolution in another study [8]. Gilboa proposed a total variation spectral approach for texture analysis and processing [9], where texture information may partially be considered as a type of noise.

In this study, we present a robust method for removal of Gaussian noise from color images based on the trace-based approach including a fidelity term that assists preservation of the input image structure. Also, we estimate the standard deviation of Gaussian noise via the wavelet-based approach.

In the following sections, we give some detailed information about additive noises and mention about removal of additive Gaussian noise from input images by employing PDEs-based method including both smoothing term and fidelity term.

2. Additive Noise Removal

A corrupted gray-level image with an additive noise is formulated as follows:

$$f = f_0 + \text{additive noise} \tag{1}$$

where $f\colon \Omega \to \mathbb{R}$ is a noisy image and $f_0\colon \Omega \to \mathbb{R}$ is an original image. Both are defined in a closed region Ω. $\mathbf{x} = (x, y)$ indicates image coordinates. Impulsive noise has the following statistical model:

$$P_X(x) = \begin{cases} probability_w, & x = \text{a near white value} \\ probability_b, & x = \text{a near black value} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where X is a random variable. Gaussian noise has the probability distribution function (PDF) shown as below:

$$P_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{3}$$

where σ^2 is the noise variance and μ is the mean value of Gaussian noise.

Types of additive noises are able to be estimated by a selected flat region of the input image (See Figs. 1a-c and 2a-c). It is depicted in Fig. 1c that salt and pepper noise is easily eliminated in the flat region by using a median filter because this noise type has two pick values in the near minimum and maximum intensities of the input image. However, this filter may blur non-flat regions, especially in image edges. In this case, energy functional based on PDEs is given for salt and pepper noise as follows [10, 11]:

$$E_1(u,f) = \int_\Omega (|\nabla u| + \lambda|f-u|)dxdy \qquad (4)$$

where $u{:}\,\Omega \to \mathbb{R}$ is an evolved (resulting) image. $\nabla \cdot = (\partial \cdot/\partial x,\ \partial \cdot/\partial y)^T$ is the gradient operator. Energy fuctional $E_1(\cdot)$ can be written based on the Φ formulation, which is a general form, as follows:

$$E_1(u,f) = \int_\Omega (\Phi(|\nabla u|) + \lambda|f-u|)dxdy \qquad (5)$$

where $\Phi(|\nabla u|) = \sqrt{1+|\nabla u|^2}$ is set for preventing the singularity while the equation is evolved [5].

If Eq. (5) is solved based on the Euler-Lagrange (E-L) method, the following Eq. (6) is obtained:

$$-\frac{\partial E_1}{\partial u} = \mathrm{div}\left(\Phi'(\cdot)\frac{\nabla u}{|\nabla u|}\right) + \lambda \mathrm{sign}(f-u) \qquad (6)$$

where $\mathrm{div}(\cdot)$ is divergence and $\mathrm{sign}(\cdot)$ is the sign function. The smoothing term is based on the total variation approach and the fidelity term uses L^1 norm due to the fact that salt and pepper noise is easily detected using the sign function. Namely, each pixel value in the input image is not corrupted by salt and pepper noise, where sign change of subtraction between the pixel value of the input image f and the evolved image u is zero. Also, if the fidelity term is not included in the energy functional, the evolved image has a constant value at the convergence ($t \to \infty$). On the other hand, Gaussian noise corrupts almost all pixel values in the input image as seen in Fig. 2c. Therefore the fidelity term uses L^2 norm in the energy functional as follows (Restored image is shown in Fig. 2d as well):

$$E_2(u,f) = \int_\Omega \left(|\nabla u| + \frac{1}{2}\lambda \times (f-u)^2\right)dxdy. \qquad (7)$$

If energy fuctional $E_2(\cdot)$ is written based on the Φ formulation,

$$E_2(u,f) = \int_\Omega \left(\Phi(|\nabla u|) + \frac{1}{2}\lambda \times (f-u)^2\right)dxdy \qquad (8)$$

is obtained. Energy fuctional $E_2(\cdot)$ is minimized by E-L method as follows:

$$-\frac{\partial E_2}{\partial u} = \mathrm{div}\left(\Phi'(\cdot)\frac{\nabla u}{|\nabla u|}\right) + \lambda \times (f-u) \qquad (9)$$

Since we are interested in Gaussian noise, we give more detailed information about Gaussian noise removal. Here, due to the fact that the mean value of f is almost equal to the mean value of u, empricial noise variance $\hat{\sigma}^2$ is updated at each iteration as $\hat{\sigma}^2 = |\Omega|^{-1}\int_\Omega (f-u)^2 dxdy$. In this case, λ is optimized as

$$\lambda = \frac{1}{|\Omega|\sigma^2}\int_\Omega \mathrm{div}\left(\Phi'(\cdot)\frac{\nabla u}{|\nabla u|}\right)(u-f)dxdy. \qquad (10)$$

When λ is converged, empricial variance $\hat{\sigma}^2$ is equal to noise variance σ^2. Note that this approach is not valid for salt and pepper noise [5].

Eq. (8) is evolved using the iterative gradient descent method as follows:

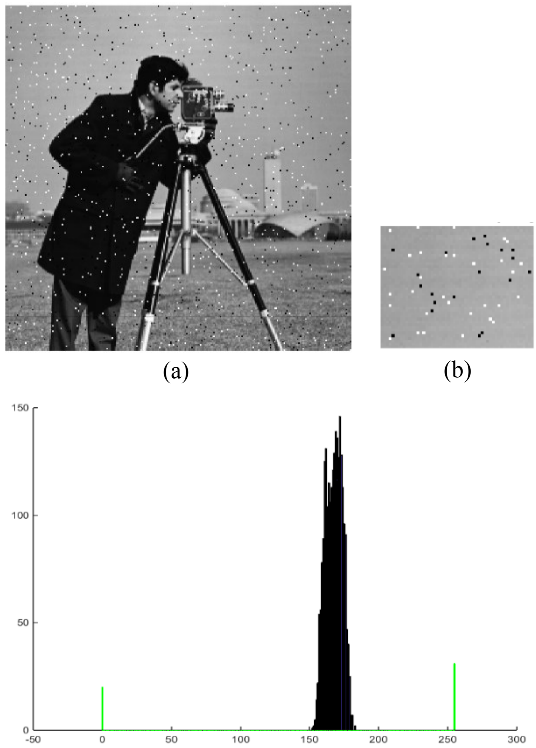
$$\begin{cases} u(t=0;\mathbf{x}) = f(\mathbf{x}) \\ \dfrac{\partial u}{\partial t} = -\dfrac{\partial E_2}{\partial u} = \mathrm{div}\left(\Phi'(\cdot)\dfrac{\nabla u}{|\nabla u|}\right) + \lambda \times (f-u) \end{cases} \qquad (11)$$

where t indicates time or scale.

Finally the following equation is given based on the explicit diffusion scheme:

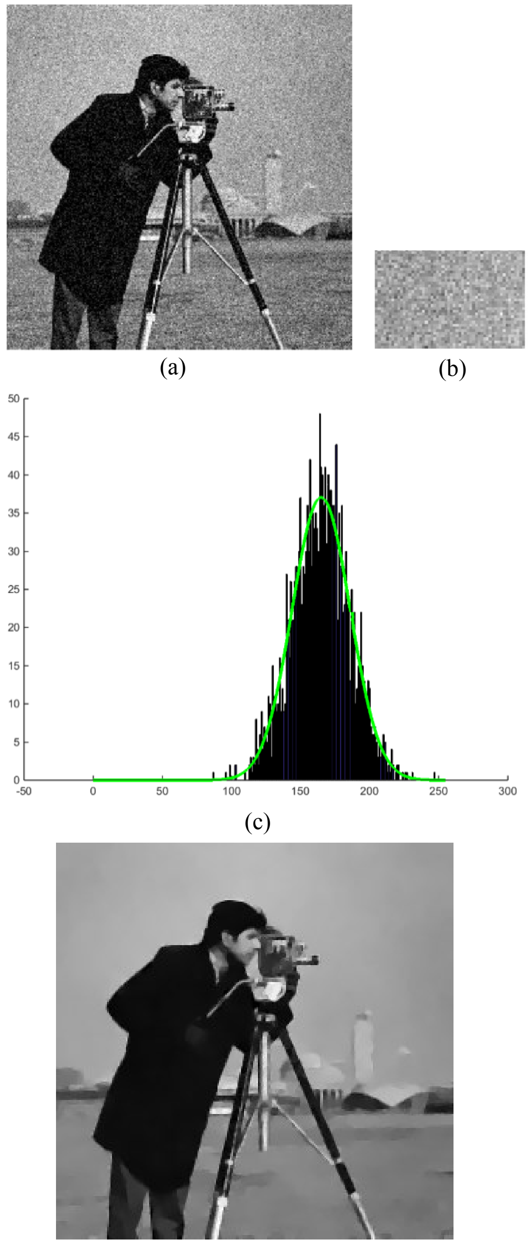
$$u^{n+1} = u^n + dt \times (\partial u/\partial t)^n \qquad (12)$$

where n iteration count and dt is time step. Here, the time step is simply set to $dt \leq 0.25$ for the CFL condition [5, 12].



(c)

Fig. 1. Image including additive salt and pepper noise with the noise density of d = 2% : a) Cameraman test image, b) selected block of flat region of the noisy image, and c) histogram (black bars) and PDF (green line) of the noisy block



(a) (b) (c) (d)

Fig. 2. Image including additive Gaussian noise with the variance of $\sigma^2 = 400$ and the mean value of $\mu = 0$: a) Cameraman test image, b) selected block of flat region of the noisy image, c) histogram (black bars) and PDF (green line) of the noisy block, and d) restored image

3. Presented Method

Let $\mathbf{f}(f_1, f_2, f_3): \Omega \to \mathbb{R}^3$, $\mathbf{f}_0: \Omega \to \mathbb{R}^3$ and $\mathbf{u}: \Omega \to \mathbb{R}^3$ represent a noisy color input image, an original color input image and an evolved color image, respectively. In the presented method, we add a fidelity term to the trace-based method [7] in order to obtain the resulting image properly.

First, in the trace-based method, the structure tensor is obtained for each pixel in the color image as follows:

$$\mathbf{G} = \sum_{i=1}^{3} \nabla u_i \nabla {u_i}^T = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \quad (13)$$

where i, $1 \le i \le 3$, indicates each channel component of the color image and $\nabla u_i = [\partial u_i/\partial x\ ,\ \ \partial u_i/\partial y]^T$. Mutually perpendicular eigenvectors $\varphi^{\pm}$ and two positive eigenvalues $\lambda^{\pm}$ related to the eigenvectors are computed from the structure tensor matrix as below:

$$\varphi^{\pm} /\!/ \left[2g_{12},\quad g_{22} - g_{11} \pm \sqrt{(g_{11} - g_{22})^2 + 4g_{12}^2}\right]^T \text{and}$$

$$\lambda^{\pm} = \left(g_{11} + g_{22} \pm \sqrt{(g_{11} - g_{22})^2 + 4g_{12}^2}\right)/2.$$

Positive eigenvalues $\lambda^{\pm}$ give the following information for the current pixel:

i. If $\lambda^+ \cong \lambda^- \cong 0$, then it may be in a flat region,
ii. If $\lambda^+ \gg \lambda^-$, then it may be in an edge region,
iii. If $\lambda^+ \cong \lambda^- \gg 0$, then it may be in a corner region.

Also, the structure tensor can be smoothed by the LPF such as Gaussian filter g_σ with standard deviation σ in order to obtain more robust results: $\mathbf{G}_\sigma = \mathbf{G} * g_\sigma$, where $*$ is the convolution operator.

Secondly, Tschumperle and Deriche [5] use the diffusion tensor in order to remove noises from color images:

$$\mathbf{T} = s^-(\lambda^+, \lambda^-)\varphi^- {\varphi^-}^T + s^+(\lambda^+, \lambda^-)\varphi^+ {\varphi^+}^T \quad (14)$$

where $s^{\pm}: \mathbb{R}^2 \to \mathbb{R}$ are two functions and are set for the noise removal as follows:

$s^-(\lambda^+, \lambda^-) = (1 + \lambda^+ + \lambda^-)^{-a_1}$ and

$s^+(\lambda^+, \lambda^-) = (1 + \lambda^+ + \lambda^-)^{-a_2}$, where $a_1 < a_2$.

If the noise removal process is done

i. only in the direction of eigenvector φ^-, it is called an anisotropic smoothing,
ii. in the flat region, it is called an isotropic smoothing. In this case, $\mathbf{T} \cong \mathbb{I}$ is obtained as a unit matrix and the trace-based method behaves as Laplace equation.

Lastly, the other process steps for the presented method are given as follows:

$$E_3 = \int_\Omega \left(\Phi(|\nabla \mathbf{u}|) + \frac{1}{2}\lambda \times (\mathbf{f} - \mathbf{u})^2\right) dxdy \quad (15)$$

$$\lambda = \frac{1}{3|\Omega|\tilde{\sigma}^2} \sum_{i=1}^{3} \int_\Omega \text{trace}(\mathbf{TH}_i)(\mathbf{u} - \mathbf{f}) dxdy \quad (16)$$

$$\begin{cases} \mathbf{u}(t=0;\mathbf{x}) = \mathbf{f}(\mathbf{x}) \\ \dfrac{\partial \mathbf{u}}{\partial t} = \operatorname{trace}(\mathbf{T}\mathbf{H}_i) + \lambda \times (\mathbf{f}-\mathbf{u}) \end{cases} \quad (17)$$

where trace(·) represents trace of the matrix and $\mathbf{H}_i$ is Hessian matrix:

$$\mathbf{H}_i = \begin{bmatrix} \partial^2 u_i/\partial x^2 & \partial^2 u_i/\partial x \partial y \\ \partial^2 u_i/\partial y \partial x & \partial^2 u_i/\partial y^2 \end{bmatrix}.$$

In the presented method, the standard deviation of Gaussian noise is computed based on the wavelet transform. After the noisy color input image is converted into a gray-level image, the wavelet transform of the image is computed. And then the standard deviation of Gaussian noise is estimated as follows:

$$\tilde{\sigma} \cong \operatorname{median}(|D|)/0.6745 \quad (18)$$

where D is the diagonal detail coefficient of the result of the wavelet transform in the first decomposition level.

In order to summarize the presented method, we depict a flowchart as in in Fig. 3.

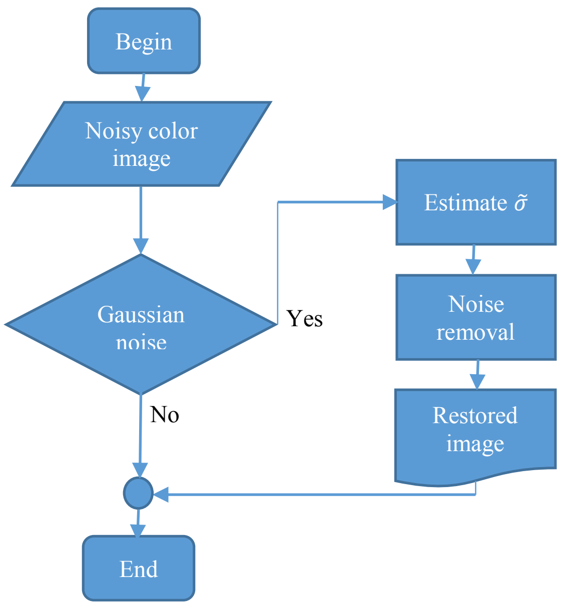


Fig. 3. Flowchart of the presented method

4. Experimental Results

We test the validity of the presented method on Lenna and Pepper color images. Both of the test images are in the size of 256×256. The artificial Gaussian noise with the standard deviation of $\sigma = 20$ and mean value of $\mu = 0$ is added to both test images. The iteration count, the standard deviation of Gaussian filter g_σ for the structure tensor smoothing, time step and diffusion limiters are respectively set to $n = 100$, $\sigma = 0.5$, $\mathrm{dt} = 0.2$, and $a_1 = 0.5$ and $a_2 = 0.9$ for the trace-based method. In addition, in the presented method, the loop is terminated based on the error, $\epsilon \leq 0.1$, as a difference between the results of the current iteration and the previous iteration.

The results obtained from the test images are depicted in Figs. 4a-d and 5a-d. Although the presented method removes fine details more compared to the trace-based method [7], better visual results of the presented method are generated than the ones of the traced-based method are generated as shown in Figs 4c-d and 5c-d. In addition to this, the quantity value, i.e. PSNR value, obtained from the presented method is better than the trace-based method. In the presented method, standard deviations are also estimated by the wavelet transform as $\tilde{\sigma} \cong 20 \pm 0.7$.

Mean square error (MSE) and peak signal to noise ratio (PSNR) are computed as follows:

$$\mathrm{MSE} = \frac{1}{3|\Omega|}\sum_{i=1}^{3}\int_\Omega \left(u_i - f_{0_i}\right)^2 dxdy$$
$$\mathrm{PSNR} = 20\log_{10}\left(\frac{255}{\mathrm{MSE}}\right) \quad (19)$$

The source code was written in Microsoft Visual C++ platform and run on a PC with an Intel® Core™ i5-3470 3.20 GHz CPU and with 8 GB of RAM.

5. Conclusions

We present a PDEs-based method for Gaussian noise removal from color images. The generated results based on the presented method seem promising. In the future study, we will extend the presented method to estimate and to remove both additive impulsive and Gaussian noises. In addition to this, the presented method can be applied on the noisy color images having only structure information. So, we will add some approaches to the presented method in order to take both structure and texture information of the noisy image into consideration.

6. References

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Fig. 4. Noise removal results for Lenna test image: a) Original image, b) corrupted image by Gaussian noise with $\sigma = 20$ and $\mu = 0$, c) result of trace-based method (PSNR = 29.65 dB) [7], and d) result of presented method (PSNR = 30.40 dB)

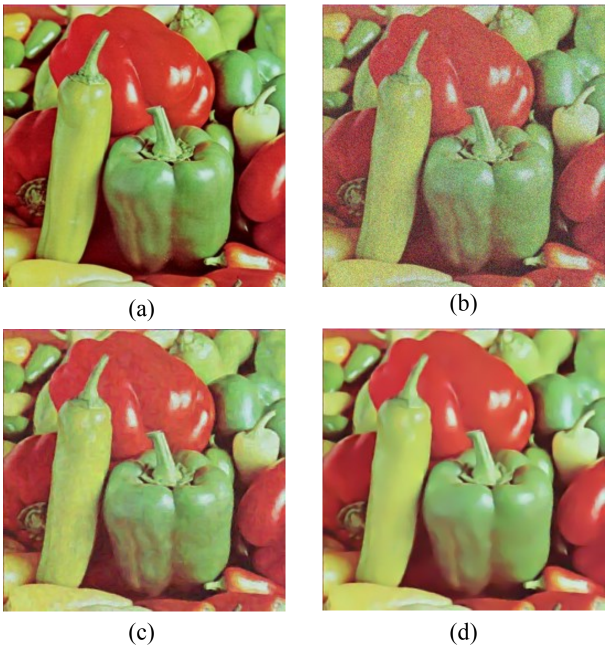

Fig. 5. Noise removal results for Pepper test image: a) Original image, b) corrupted image by Gaussian noise with $\sigma = 20$ and $\mu = 0$, c) result of trace-based method (PSNR = 29.71 dB) [7], and d) result of presented method (PSNR = 30.57 dB)