

Identification of Fractional Order Model Structures Using Conversion of Infinite Terms of a Response to Finite Terms

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Abstract

In this paper, a new method is proposed to determine parameters of some types of fractional order models to represent a system dynamics using its step response. The model structure is assumed to have rational commensurate order as $\alpha = M/N$. By this assumption infinite terms of the model step response is converted into finite terms limited to N . The proposed method is practically applicable on all fractional order model structures due to the order approximation imposed by the limited precisions in available computational tools.

1. Introduction

Fractional calculus is an old mathematical topic which has been discussed by some researchers such as [1] and recently has been used to describe some physical phenomena reported in the processes like electromagnetic [2], electromechanic [3], heat transfer [4], biology [5], and so on.

Identification of fractional order systems is a complicated task since one needs to determine both fractional orders and model parameters. Also due to the infinite memory property, it would be difficult to estimate order of these systems from their time responses. In this paper a new identification method in time domain is proposed which is based on approximation by rational order transfer functions. The proposed method eliminate necessity of truncation in the calculation of the fractional order models output which is the main source of the approximation error usually encountered in the simulation of the fractional order models.

2. Conversion of Infinite Terms of a Response to Finite Terms

We study a special case of fractional order model structures that have rational commensurate order as $\alpha = M/N$. In practice, when identification is accomplished using computational tools with limited accuracy, orders would be rational and thus the present discussion would be helpful for such applications.

2.1. Simple Pseudo Pole

First we consider a single pseudo pole fractional order model structure which is described as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s^\alpha - p} \quad (1)$$

This can equivalently be written as

$$G(s) = \frac{Ks^{-\alpha}}{1 - ps^{-\alpha}} = Ks^{-\alpha} \sum_{k=0}^{\infty} (ps^{-\alpha})^k \quad (2)$$

The summation part of (2) is expanded as below

$$\begin{aligned} \sum_{k=0}^{\infty} (ps^{-\alpha})^k &= \sum_{k=0}^{N-1} (ps^{-\alpha})^k + \sum_{k=N}^{2N-1} (ps^{-\alpha})^k + \dots \\ &= \sum_{k=0}^{N-1} (ps^{-\alpha})^k + p^N s^{-N\alpha} \sum_{k=0}^{N-1} (ps^{-\alpha})^k + \dots \\ &= \sum_{k=0}^{N-1} (ps^{-\alpha})^k / (1 - p^N s^{-M}) \end{aligned} \quad (3)$$

By defining $R(s)$ as follows

$$R(s) \triangleq K \sum_{k=1}^N p^{k-1} s^{-\alpha k} \quad (4)$$

one can write

$$G(s) = \frac{R(s)}{1 - p^N s^{-M}} \quad (5)$$

2.2. Double Pseudo Poles

Let us know consider the following model structure

$$G(s) = \frac{m_1 s^\alpha + m_0}{n_2 s^{2\alpha} + n_1 s^\alpha + 1} \quad (6)$$

From partial fraction expansion one can write

$$G(s) = \frac{K_1}{s^\alpha - p_1} + \frac{K_2}{s^\alpha - p_2} \quad (7)$$

Similar to the case of single pseudo pole, (7) can be written as

$$G(s) = \frac{R_1(s)}{1 - p_1^N s^{-M}} + \frac{R_2(s)}{1 - p_2^N s^{-M}} \quad (8)$$

where

$$R_1(s) \triangleq K_1 \sum_{k=1}^N p_1^{k-1} s^{-\alpha k}, R_2(s) \triangleq K_2 \sum_{k=1}^N p_2^{k-1} s^{-\alpha k} \quad (9)$$

Above mentioned equations can be simply extended to any

commensurate fractional order model structures.

3. Identification Procedure

Time domain identification using step response is one of the commonly implemented methods especially in industrial processes. This method has been discussed in some works like [6], [7], and [8]. Furthermore, the step response model has been widely used to tune parameters of integer or fractional order PIDs for example using Ziegler-Nichols rules.

The proposed identification procedure is discussed for different common cases in the following subsections.

3.1. Single Pseudo Pole Fractional Model with Known Order

Different signals can be considered as the exciting input in the proposed time domain identification. We use the unit step input at the moment and consider a simple pseudo pole fractional model structure as (1). Thus from (5), one can write

$$Y(s) = \frac{s^{-1}R(s)}{1-p^N s^{-M}} \quad (10)$$

or equivalently,

$$Y(s)(1-p^N s^{-M}) = s^{-1}R(s). \quad (11)$$

Multiplying (11), by s^{-I} and applying (4), result in

$$s^{-I}Y(s) - p^N s^{-(M+I)}Y(s) = K \sum_{k=1}^N p^{k-1} s^{-(\alpha k + I)}. \quad (12)$$

Using inverse Laplace transform one obtains

$$y^{(-I)}(t) = \sum_{k=1}^N \frac{K p^{k-1} t^{(\alpha k + I)}}{\Gamma(\alpha k + I + 1)} + p^N y^{(-M-I)}(t) \quad (13)$$

where $y^{(-I)}$ is the I^{th} time integral of y from zero to t and could be determined for instance using the trapezoidal numerical integration method.

Now one can apply the least squares estimation approach to the following linear regression model

$$y^{(-I)}(t) = \phi^T(t)\theta + \varepsilon(t) \quad (14)$$

where

$$\phi^T(t) = \left[\frac{t^{(\alpha+I)}}{\Gamma(\alpha+I+1)}, \dots, \frac{t^{(N\alpha+I)}}{\Gamma(N\alpha+I+1)}, y^{(-M-I)} \right], \quad (15)$$

$$\theta^T = [K, Kp, \dots, Kp^{N-1}, p^N].$$

In (14), $y^{(-I)}$ is linear with respect to parameters θ , though the model parameters K and p are appeared nonlinearly in θ . When θ is determined, the model parameters K and p are calculated through a procedure which is explained later.

Let choose J points from the unit step response of the process where $J > N$. Then construct matrix Φ and vector Υ from the selected points and formulae in relations (13) and (15).

$$\Phi \triangleq \begin{bmatrix} \phi^T(t_1) \\ \vdots \\ \phi^T(t_J) \end{bmatrix}, \quad \Upsilon \triangleq \begin{bmatrix} y^{(-I)}(t_1) \\ \vdots \\ y^{(-I)}(t_J) \end{bmatrix}. \quad (16)$$

The following cost function is minimized to determine θ .

$$V = \frac{1}{2}(\Upsilon - \Phi^T \theta)^T (\Upsilon - \Phi^T \theta). \quad (17)$$

Consequently the unknown parameters θ are obtained as below

$$\theta = (\Phi^T \Phi)^{-1} \Phi^T \Upsilon. \quad (18)$$

Notice that the uniqueness of the solution relates to non singularity of matrix $\Phi^T \Phi$.

3.2. Single Pseudo Pole Fractional Model with Unknown Order

When the order of the fractional system is unknown, performing an optimization process is necessary to approximate an appropriate model for the true system. The related stages could be described as follow

- S₁. First select an integer number N (larger N improves the accuracy in the expense of computational requirement).
- S₂. The optimization algorithm is performed for different values of $M \in [1, 2N]$ (according to the commensurate order's stability range).
- S₃. Order α is approximated by a rational number M/N .
- S₄. According to some specified criteria like minimizing V in (17), the best model is chosen from $2N$ calculated models.
- S₅. If none of the calculated model has desired accuracy, then increase value of N (this is equivalent to decrease α) and return to S₂.

Nazarian and his colleagues showed that the computer's round off errors would limit the process for smaller values of commensurate order α [9].

3.3. Double Pseudo Pole Fractional Model with Known Order

Let the model structure be as the one discussed in (6) to (9)

$$G(s) = \frac{R_1(s)}{1-p_1^N s^{-M}} + \frac{R_2(s)}{1-p_2^N s^{-M}} \quad (19)$$

Applying the unit step input to the system and performing the same calculation described in subsection 3.1 result in (14) with the following ϕ and θ .

$$\phi^T(t) = [\phi_1^T(t), \phi_2^T(t), \phi_3^T(t)], \quad \theta^T = [\theta_1^T, \theta_2^T, \theta_3^T], \quad (20)$$

$$\phi_1^T(t) = \left[\frac{t^{(\alpha+I)}}{\Gamma(\alpha+I+1)}, \dots, \frac{t^{(N\alpha+I)}}{\Gamma(N\alpha+I+1)} \right],$$

$$\phi_2^T(t) = \left[\frac{-t^{(\alpha+I+M)}}{\Gamma(\alpha+I+M)}, \dots, \frac{-t^{(N\alpha+I+M)}}{\Gamma(N\alpha+I+M)} \right], \quad (21)$$

$$\phi_3^T(t) = [y^{(-M-I)} \quad -y^{(-2M-I)}],$$

and

$$\begin{aligned} \theta_1^T &= [K_1 + K_2, K_1 p_1 + K_2 p_2, \dots, K_1 p_1^{N-1} + K_2 p_2^{N-1}], \\ \theta_2^T &= [K_1 p_2^N + K_2 p_1^N, \dots, K_1 p_2^N p_1^{N-1} + K_2 p_1^N p_2^{N-1}], \\ \theta_3^T &= [p_1^N + p_2^N, p_1^N p_2^N]. \end{aligned} \quad (22)$$

The unknown parameters θ are determined from (18). p_1^N and p_2^N can be determined from the last relation in (22). Then K_1 and K_2 are calculated from obtained values of θ_1 and θ_2 . Since there are redundancies in calculating K_1 and K_2 from θ_1 and θ_2 , one may use all rows of θ_1 and θ_2 along with an optimization algorithm to determine the most proper values for K_1 and K_2 .

In subsequent sections the proposed method has been applied to estimate unknown parameters of the mentioned fractional model structures using the unit step response of a given system.

4. Single Pseudo Pole Examples

Consider the following stable fractional order system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s^\alpha - p} = \frac{3}{s^{1.6} + 2} \quad (23)$$

4.1. Noise Free Case

Let assume that the fractional order α is unknown and the order accuracy is selected as 0.2. Therefore N should be 5 and M would change from 1 to 10. The unit step response of the system is calculated using Mittag-Leffler function evaluated with accuracy $1e^{-4}$ (Fig. 1). 1000 points of the step response selected logarithmically between 0.1 to 10 seconds were used to estimate the unknown parameters. Geometrically spaced samples make less numerical problems in calculation of $(\Phi^T \Phi)^{-1}$.

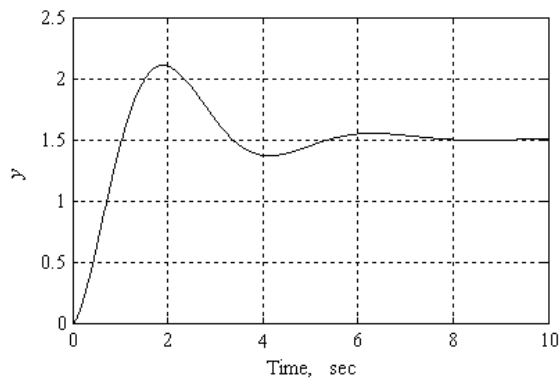


Fig. 1. The unit step response of the system (23).

The integration order I is set to zero. Tables 1 and 2 show the obtained results. Here, only two first rows of the estimated θ were used to calculate the model parameters K and p . Any other choice of two rows of θ results in almost similar values for these parameters.

4.2. Adding Gaussian Noise

As the noisy case, a random number generated by a normal

distribution function with zero mean and variance 0.01 has been added to each calculated output sample. The resulted output signal is depicted in Fig. 2. The same estimation procedure as in the previous case was carried out here. The estimated parameters are listed in Table 3.

Table 1. Noise free estimation for different orders.

M	N	$\alpha = M / N$	V	K	$-p$
1	5	0.2	15.11	49.4	3.95
2	5	0.4	8.40	-3.59	4.07
3	5	0.6	4.04	-0.94	4.58
4	5	0.8	98.64	-0.05	98.64
5	5	1.0	2.29	0.74	-3.73
6	5	1.2	13.60	2.18	0.92
7	5	1.4	2.67	2.20	0.75
8	5	1.6	1.2e-6	3.00	2.00
9	5	1.8	56.02	2.27	1.64
10	5	2.0	166	1.74	1.50

Table 2. Noise free estimated θ for $M = 8, N = 5$.

θ	Values
K	2.9999
Kp	-5.9994
Kp^2	11.9978
Kp^3	-23.9933
Kp^4	47.9915
p^5	-31.9945

Table 3. Noisy case estimation for different orders.

M	N	$\alpha = M / N$	V	K	$-p$
1	5	0.2	23.71	49.4	3.93
2	5	0.4	16.79	-3.36	4.15
3	5	0.6	13.17	-0.87	4.84
4	5	0.8	34.45	0.11	-42.42
5	5	1.0	9.87	0.78	-3.32
6	5	1.2	20.06	2.17	0.92
7	5	1.4	10.73	2.19	0.76
8	5	1.6	8.49	2.98	1.99
9	5	1.8	64.08	2.26	1.64
10	5	2.0	172.5	1.73	1.50

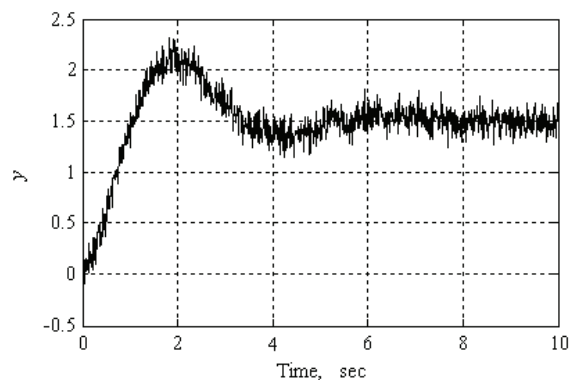


Fig. 2. The unit step response of the system (23) with the added Gaussian noise.

In either case results presented in the eighth row are corresponded to the minimum estimation error and the estimated

parameters are very close to the actual ones. In the following subsection, effect of integration order I is investigated.

4.3. Effect of the Integration Order I

Since output of the system is measured in real applications, it contains measurement noise. On the other hand, since the measurement noise concentrates usually in high frequencies, integration of y , effectively attenuates the noise effect and so, whatever large I is selected, the noise is more reduced in $y^{(-I)}$ and $y^{(-I-M)}$. Of course, very large $I + M$ generates noticeable integration errors and therefore smaller integration steps will be necessary in such cases.

Table 4 indicates the estimated parameters of the above system in the noisy case for different values of I . As expected, enlarging the integration order provides better estimation result. The order estimation was performed by the previously mentioned procedure.

Table 4. Noisy case parameter estimation for different integration order I .

I	V	K	P
0	8.49	2.9785	1.9938
1	0.017	2.9880	1.9969
2	0.004	2.9944	2.0002

4.4. Parameter Estimation for an unstable system

The proposed method in this paper has capability of fitting a model structure to an unstable system in an open loop in a similar way discussed above. In this case one should apply the procedure only to the initial data before the output of the system grows too much. For instance, consider the following unstable fraction order system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s^\alpha - p} = \frac{3}{s^{1.6} - 2} \quad (24)$$

Similar noise signal as discussed in subsection 4.2 with different variances was considered here. Fig. 3 shows the output for a specific noise level. The 1000 selected output samples were chosen from 0.1 to 1 seconds of the simulation time.

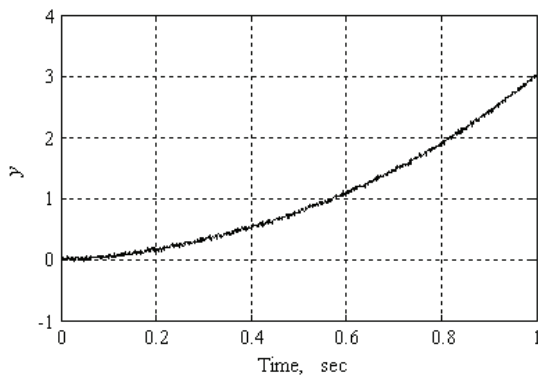


Fig. 3. Simulated step response with added noise for the unstable system (24).

Table 5 indicates the parameter estimation results in the noise free and noisy cases, after estimating the order by the previously

mentioned procedure. The numerical problems due to the badly scaled matrix Φ have direct relation to the time response values and can be resolved by the known existing improvement methods. The results shown in Table 5 have been obtained by pseudo inverse method.

Table 5. Noise free and noisy case parameter estimation for the unstable system (24).

Noise var.	V	K	P
0	1.1e-13	3.00	2.00
0.01	0.85	2.98	2.08
0.02	0.34	2.96	2.15

5. Double Pseudo Poles Examples

In this section the proposed method is examined in the identification of double pseudo poles model structures with real and complex poles values.

5.1. Real Pseudo Poles

The following stable fractional order system with two real pseudo poles is considered.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^{1.5} + 4}{s^3 + 3s^{1.5} + 2} = \frac{3}{s^{1.5} + 1} - \frac{2}{s^{1.5} + 2} \quad (25)$$

The order accuracy is chosen as 0.5. Thus N should be 2, and therefore, M would belong to $\{1, 2, 3, 4\}$. The integration order I was set to zero and a Gaussian noise was added to the output. Table 6 indicates the results for $M = 3$ in the noise free and noisy cases with different variances.

Parameters p_1 and p_2 were determined from last two rows of Table 6. The other two unknown parameters K_1 and K_2 were calculated from two first rows of Table 6. It should be noted that one can obtain the four unknown parameters of the model using all rows of Table 6 and performing a nonlinear programming. In that case results would be slightly different from those calculated here.

Table 6. Estimation results for a two real pseudo poles system (25).

Noise var.	0	0.0049	0.01
$K_1 + K_2$	1.0000	0.9814	0.9737
$K_1 p_1 + K_2 p_2$	0.9998	1.0135	1.0184
$K_1 p_1^2 + K_2 p_1^2$	10.0002	9.9569	9.9336
$K_1 p_2^2 p_1 + K_2 p_1^2 p_2$	-8.0004	-7.9624	-7.9424
$p_1^2 + p_2^2$	5.0003	4.9842	4.9751
$p_1^2 p_2^2$	4.0002	3.9885	3.9816

Table 7. Parameter estimation results for a two real pseudo poles system (25).

Noise var.	0	0.0049	0.01
K_1	-1.9996	-2.0056	-2.0090
K_2	2.9996	2.9870	2.9827
p_1	-2.0001	-1.9957	-1.9932
p_2	-1.0000	-1.0007	-1.0011

There would be two solutions for either p_1 or p_2 if one uses the last two rows of Table 6. The negative values are selected here since they satisfy the two middle rows of Table 6. The final results for the unknown parameters K_1 , K_2 , p_1 , and p_2 are seen in Table 7. As it is expected the estimation error increases by reducing the signal to noise ratio. However, the estimation results are good enough for the given simulation scenarios.

5.2. Complex Pseudo Poles

As the last simulation example a stable fractional order system with the following transfer function is considered. This system has two complex conjugate pseudo poles.

$$G(s) = \frac{3}{s^3 + 2s^{1.5} + 1.25} = \frac{3j}{s^{1.5} + 1 + 0.5j} - \frac{3j}{s^{1.5} + 1 - 0.5j} \quad (26)$$

We fit a double pseudo poles fraction model structure to the unit step response of the system. The order accuracy is selected as 0.5. Thus N has to be 2, and therefore M would admit values between 1 and 4. Again we set the integration order I to zero and added a Gaussian noise signal with different variances to the simulated unit step response of the system. Table 8 indicates the estimation results for $M = 3$ in the noise free and noisy cases.

We have performed the same procedure as in the previous case to calculate the unknown parameters K_1 , K_2 , p_1 , and p_2 (Table 9). Similar to the obtained results in the previous subsections, the estimation error is not noticeable even in the noisy cases. Because of the noise and computational round off effect, the calculated values for parameters K_1 and K_2 have small real parts that are negligible.

Table 8. Estimation results for a two complex pseudo poles system (26).

Noise var.	0	0.0049	0.01
$K_1 + K_2$	0.0000	-0.0173	-0.0247
$K_1 p_1 + K_2 p_2$	3.0000	3.0339	3.0484
$K_1 p_2^2 + K_2 p_1^2$	6.0001	6.0521	6.0741
$K_1 p_2^2 p_1 + K_2 p_1^2 p_2$	-3.7501	-3.7763	-3.7873
$p_1^2 + p_2^2$	1.5001	1.5199	1.5282
$p_1^2 p_2^2$	1.5626	1.5750	1.5803

Table 9. Parameter estimation results for a two complex pseudo poles system (26).

Noise var.	0	0.0049	0.01
K_1	3.0j	-0.0086+3.0729j	-0.0123+3.0950j
K_2	-3.0j	-0.0087-3.0729j	-0.0124-3.0950j
p_1	-1.0-0.5j	-1.0037-0.4975j	-1.0053-0.4965j
p_2	-1.0+0.5i	-1.0037+0.4975j	-1.0053+0.4965j

6. Conclusions

In this paper, a new method has been proposed to estimate unknown parameters of one or two pseudo pole fractional order model structures. The proposed method is extendable with some

complexity to the commensurate order transfer functions with any number of pseudo poles.

The main shortcoming of the proposed method is probability of ill conditioning of matrix $\Phi^T \Phi$ which becomes more serious when the order accuracy ($1/N$) is small. The problem is magnified when the measurement noise or round off error level is not negligible. Use of the conditioning methods would alleviate the setback in some extent, however, enlarging the order accuracy would be the more effective remedy in this regard. Of course this option would squeeze the model set and reduce its probability to include the true system.

6. References

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