CRAMÉR-RAO BOUNDS FOR THE NEAR-FIELD SOURCE PARAMETERS

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ABSTRACT

The performance of the unconditional maximum likelihood location estimator for the near-field sources is studied based on the concentrated likelihood approach to abtain CRB. Some insights into the achievable performance of the conditional maximum likelihood algorithm is obtained by numerical evaluation of the Cramér-Rao bounds for different test cases.

1. INTRODUCTION

The problem of localizing multiple narrow band sources by a passive sensor array is common to diverse applications including radar, sonar, communication, seismology and electronic surveillance [1]. However, majority of the localization methods deals with the case in which the source is assumed to be in the far-field of the array [1], [2]. That is, the source is assumed to be at an infinite distance from the array, and hence, the waves emitted by the sources can be considered as plane waves. Thus each source location can be characterized by only the azimuth (bearing). When the sources are located close to the array (i.e., near-field), the inherent curvature of the waveforms can no longer be neglected. Therefore, the spherical wavefronts in the near-field scenario must be considered and the location of each source have to be parametrized in terms of the direction of arrival (DOA) and range [3], [4], [5], [6].

Regarding the assumption on the narrow-band source signals, there are two different types of models. These two models lead corresponding ML solutions. The models are

i. Conditional model (CM) which assumes the signals to be unknown but deterministic (i.e., the same in all realizations)

ii. Unconditional Model (UM) which assumes the signals to be random.

ML methods corresponding to the signal models (i) and (ii) are termed conditional ML and unconditional ML respectively. Expectation/Maximization (EM) based conditional ML (signal model (i)) near-field location estimator have been studied in [5].

The asymptotical performance of an unconditional ML near-field localization technique is analysed via the Cramér Rao Bounds (CRB) provides benchmarks for evaluating the performance of actual estimators. The technique for the derivation of CRB's used here in relies as modulators of the log-likelihood function by replacing the nuisance parameters with their ML estimates. It therefore avoids the process of explicitly calculating and invert the entire FIM. We substitute the ML estimates of the array observation covariance matrix to obtain concentrated covariance matrix. We then calculated the CRB's for near-field source location parameters.

We use the standard narrowband observation for d near-field sources impinging on an array of M sensors [3]. Letting $\boldsymbol{\mu} \in \mathbb{R}^{d\times 1}$ and $\boldsymbol{\zeta} \in \mathbb{R}^{d\times 1}$ denote the vectors of near-field source location parameters to be estimated, M sensor outputs $\mathbf{x}(t_n) = [x_{k_{min}}(t_n), \cdots, x_{k_{max}}(t_n)]^T$, can be written in matrix form as

$$\mathbf{x}(t_n) = \mathbf{A}(\boldsymbol{\mu},\boldsymbol{\zeta})\mathbf{s}(t_n) + \mathbf{n}(t_n), \quad 1 \leq t_n \leq N \quad (1)$$

where $\mathbf{s} \in \mathbb{C}^{d\times 1}$ is the source signal vector, $\boldsymbol{s} = [\mathbf{s}^T(1), \cdots, \mathbf{s}^T(N)]^T \in \mathbb{C}^{Nd\times 1}$, $\mathbf{A}(\boldsymbol{\mu},\boldsymbol{\zeta}) = [\mathbf{a}(\mu_1,\zeta_1), \cdots, \mathbf{a}(\mu_d,\zeta_d)]$

is the array steering matrix in the near-field case which is known as a function of unknown set of parameters $\{\boldsymbol{\mu},\boldsymbol{\zeta}\}$ and $\mathbf{a}(\mu_i,\zeta_i)$ is the i^{th} array steering vector in the following form

$$\mathbf{a}(\mu_i,\zeta_i) = \begin{bmatrix} e^{j(k_{min}\mu_i + k_{min}^2\zeta_i)} \\ \vdots \\ 1 \\ e^{j(\mu_i+\zeta_i)} \\ e^{j(2\mu_i+4\zeta_i)} \\ \vdots \\ e^{j(k_{max}\mu_i + k_{max}^2\zeta_i)} \end{bmatrix}. \quad (2)$$

The following assumptions are imposed on (1):

AS1: The source signal $\mathbf{s}(t_n)$ is temporally and spatially uncorrelated circular complex Gaussian random process with zero-mean and nonsingular unknown covariance matrix $\mathbf{K_s}$,

$$\begin{aligned} E\left[\mathbf{s}(t_n)\mathbf{s}^H(t_m)\right] &= \mathbf{K_s}\delta_{t_n,t_m} \\ E\left[\mathbf{s}(t_n)\mathbf{s}^T(t_m)\right] &= \mathbf{0} \quad \text{for all } t_n \text{ and } t_m \ . \end{aligned} \quad (3)$$

where δ_{t_n,t_m} is the Kronecker delta ($\delta_{t_n,t_m} = 1$ if $t_n = t_m$ and 0 otherwise), $(\cdot)^H$ is the conjugate transpose and $(\cdot)^T$ is the transpose of a matrix.

AS2: The additive noise vector $\mathbf{n}(t_n)$ is temporally and spatially uncorrelated circular complex Gaussian process with zero-mean and standart derivative σ^2 as

$$E\left[\mathbf{n}(t_n)\mathbf{n}^H(t_m)\right] = \sigma^2\mathbf{I}\delta_{t_n,t_m} \quad (4)$$

$$E\left[\mathbf{n}(t_n)\mathbf{n}^T(t_m)\right] = \mathbf{0} \quad \text{for all } t_n \text{ and } t_m \ . \quad (5)$$

AS3: The source signal $\mathbf{s}(t_n)$ and the noise $\mathbf{n}(t_m)$ are uncorrelated for all t_n and t_m.

Based on the assumptions **AS2** and **AS3**, the array observations $\mathbf{x}$ are Gaussian distributed with zero-mean and covariance $\mathbf{K_x}(\boldsymbol{\mu}, \boldsymbol{\zeta}, \mathbf{K_s})$, i.e,

$$\begin{aligned} \mathbf{K_x}(\boldsymbol{\mu}, \boldsymbol{\zeta}, \mathbf{K_s}) &= E[\mathbf{x}(t_n)\mathbf{x}^H(t_m)] \quad (6) \\ &= \mathbf{A}(\boldsymbol{\mu}, \zeta)\mathbf{K_s}\mathbf{A}^H(\boldsymbol{\mu}, \zeta) + \sigma^2\mathbf{I} \ . \end{aligned}$$

Then joint probability density function of the observation $\boldsymbol{x} = \{\mathbf{x}(t_n), \quad t_n = 1, \cdots, N\}$ given $\{\boldsymbol{\mu}, \boldsymbol{\zeta}, \mathbf{K_s}\}$ can be written as follows:

$$\begin{aligned} f(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\zeta}, \mathbf{K_s}) &= \prod_{t_n=1}^{N} 2\pi^{-M/2}(\det \mathbf{K_x})^{-1/2} \quad (7) \\ &\times \ \exp\left(-\frac{1}{2}\mathbf{x}^H(t_n)\mathbf{K_x}^{-1}\mathbf{x}(t_n)\right) . \end{aligned}$$

The joint probability function (7) can also be written as

$$\begin{aligned} f(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\zeta}, \mathbf{K_s}) &= 2\pi^{-NM/2}(\det \mathbf{K_x})^{-N/2} \quad (8) \\ &\times \exp\left(-\frac{1}{2}\text{tr}\left[\mathbf{K_x}^{-1}\sum_{t_n=1}^{N}\mathbf{x}(t_n)\mathbf{x}^H(t_n)\right]\right) . \end{aligned}$$

where tr is the trace. The negative log-likelihood function (after discarding unnecessary terms) is

$$\begin{aligned} \mathcal{L}(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\zeta}, \mathbf{K_s}) &= -\ln\det(\mathbf{K_x}) - \quad (9) \\ &- \ \frac{1}{N}\text{tr}\left[\mathbf{K_x}^{-1}\sum_{t_n=1}^{N}\mathbf{x}(t_n)\mathbf{x}^H(t_n)\right] . \end{aligned}$$

AS2 implies that, by the law of large numbers $\mathbf{x}(t_n)$ is second-order ergodic, i.e.,

$$\mathbf{K_x} = \lim_{N\to\infty}\widehat{\mathbf{K}}_\mathbf{x} = \lim_{N\to\infty}\frac{1}{N}\sum_{t_n=1}^{N}\mathbf{x}(t_n)\mathbf{x}^H(t_n) \quad (10)$$

where $\widehat{\mathbf{K}}_\mathbf{x}$ is the sample covariance matrix. Then the negative log-likelihood function becomes

$$\mathcal{L}(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{\zeta}, \mathbf{K_s}) = -\ln\det(\mathbf{K_x}) - \text{tr}\left[\mathbf{K_x}^{-1}\widehat{\mathbf{K}}_\mathbf{x}\right] \ . \quad (11)$$

Then, the ML estimates of $\{\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\zeta}}\}$ and $\hat{\boldsymbol{s}}$ are those which locally minimizes the negative log-likelihood function (9).

2. CRAMÉR RAO BOUNDS

The *CRB* provides a lower bound on the error variance of any unbiased estimators. In particular, it provides an asymptotic near-field source location estimator. The parameter of interest is $\boldsymbol{\tau} = [\boldsymbol{\mu}^T \ \boldsymbol{\zeta}^T]^T$. To focus on the parameters of interest, we shall use a concentrated likelihood approach to obtain the *CRB* [9]. Then the ijtth element of the Fisher Information Matrix is given by

$$\mathbf{J}_{ij}(\boldsymbol{\tau}) = \mathbf{S}N\text{tr}\left(\mathbf{K_x}^{-1}\left[\frac{\partial\widetilde{\mathbf{K}}_\mathbf{x}}{\partial\tau_i}\right]_\infty \mathbf{K_x}^{-1}\left[\frac{\partial\widetilde{\mathbf{K}}_\mathbf{x}}{\partial\tau_j}\right]_\infty\right) \quad (12)$$

where $[\cdot]_\infty$ is defined as the almost sure (a.s.) limit of $[\cdot]$, and $\widetilde{\mathbf{K}}_\mathbf{x}$ is the concentrated covariance after substituting the ML estimates $\widehat{\mathbf{K}}_\mathbf{x}$ for $\mathbf{K_x}$.

The ML estimate of $\mathbf{K_s}$ can be obtained as

$$\widehat{\mathbf{K}}_\mathbf{s} = [\mathbf{A}^H\mathbf{A}]^{-1}\mathbf{A}^H\widehat{\mathbf{K}}_\mathbf{x}\mathbf{A}[\mathbf{A}^H\mathbf{A}]^{-1} - \sigma^2[\mathbf{A}^H\mathbf{A}]^{-1} \quad (13)$$

where we have suppressed the dependence of $\mathbf{A}$ on $(\boldsymbol{\mu}, \boldsymbol{\zeta})$. It is well known that $\widehat{\mathbf{K}}_\mathbf{s} \to \mathbf{K_s}$ almost surely under mild conditions. Concentrating the covariance $\mathbf{K_x}$ with respect to $\mathbf{K_s}$ yields

$$\widetilde{\mathbf{K}}_\mathbf{x} = \mathbf{A}\widehat{\mathbf{K}}_\mathbf{s}\mathbf{A}^H + \sigma^2\mathbf{I} = \mathbf{\Pi}\widehat{\mathbf{K}}_\mathbf{x}\mathbf{\Pi} + \sigma^2\mathbf{\Pi}^c \quad (14)$$

where

$$\mathbf{\Pi} \triangleq \mathbf{A}[\mathbf{A}^H\mathbf{A}]^{-1}\mathbf{A}^H \quad \text{and} \quad \mathbf{\Pi}^c \triangleq \mathbf{I} - \mathbf{\Pi} \ . \quad (15)$$

Furthermore, letting $\partial\mathbf{\Pi}/\partial\tau_i$ be denoted by $\mathbf{\Pi}_i$ for any $i = 1, \cdots, 2d$ the following can be obtained

$$\left(\frac{\partial\widetilde{\mathbf{K}}_\mathbf{x}}{\partial\tau_i}\right) = \mathbf{\Pi}_i\widehat{\mathbf{K}}_\mathbf{x}\mathbf{\Pi} + \mathbf{\Pi}\widehat{\mathbf{K}}_\mathbf{x}\mathbf{\Pi}_i + \sigma^2\mathbf{\Pi}_i^c \ . \quad (16)$$

Using the properties of the projection matrix the followings can be obtained

$$\mathbf{\Pi}^c\mathbf{\Pi} = \mathbf{0} \quad (17)$$

$$\mathbf{K_x}\mathbf{\Pi} = \mathbf{\Pi}\mathbf{K_x} \quad (18)$$

$$\text{tr}[\mathbf{\Pi}_i^c\mathbf{\Pi}^c] = \frac{1}{2}\text{tr}[(\mathbf{\Pi}^c)_i^2] = \frac{1}{2}\left(\text{tr}[\mathbf{\Pi}^c]\right)_i = \mathbf{0} \ . \quad (19)$$

Thus, taking the limit $N \to \infty$ of (16)

$$\left(\frac{\partial\widetilde{\mathbf{K}}_\mathbf{x}}{\partial\tau_i}\right)_\infty = \mathbf{\Pi}_i\mathbf{K_x}\mathbf{\Pi} + \mathbf{\Pi}\mathbf{K_x}\mathbf{\Pi}_i + \sigma^2\mathbf{\Pi}_i^c \quad (20)$$

and applying (12) give

$$\begin{aligned}\mathbf{J}_{ij}(\boldsymbol{\tau}) &= \mathbf{S}N\mathrm{tr}\big[\mathbf{K}_\mathbf{x}^{-1}(\mathbf{\Pi}_i\mathbf{K}_\mathbf{x}\mathbf{\Pi}+\mathbf{\Pi}\mathbf{K}_\mathbf{x}\mathbf{\Pi}_i+\sigma^2\mathbf{\Pi}_i^c)\\ &\times\ \mathbf{K}_\mathbf{x}^{-1}(\mathbf{\Pi}_j\mathbf{K}_\mathbf{x}\mathbf{\Pi}+\mathbf{\Pi}\mathbf{K}_\mathbf{x}\mathbf{\Pi}_j+\sigma^2\mathbf{\Pi}_j^c)\big] \quad (21)\end{aligned}$$

Next, we note that

$$\mathbf{\Pi}_i=\mathbf{\Pi}^c\mathbf{A}_i\mathbf{A}^\dagger+(\mathbf{A}^H)^\dagger\mathbf{A}_i^H\mathbf{\Pi}^c \qquad (22)$$

where $(\cdot)^\dagger$ is the pseudo inverse of $(\cdot)$. If we evaluate (21), we would obtain

$$\begin{aligned}\mathbf{J}_{ij}(\boldsymbol{\tau}) &= 2\mathbf{S}N\mathrm{Re}\Big\{\mathrm{tr}\Big\{\mathbf{A}_i^H\mathbf{\Pi}^c\mathbf{A}_j \qquad (23)\\ &\times\ \Big(\frac{\mathbf{K_s}}{\sigma^2}-(\mathbf{A}^H\mathbf{A})^{-1}+\sigma^2\mathbf{A}^\dagger\mathbf{K}_\mathbf{x}^{-1}(\mathbf{A}^\dagger)^H\Big)\Big\}\Big\}\ .\end{aligned}$$

Here,

$$\frac{\mathbf{K_s}}{\sigma^2}-(\mathbf{A}^H\mathbf{A})^{-1}+\sigma^2\mathbf{A}^\dagger\mathbf{K}_\mathbf{x}^{-1}(\mathbf{A}^\dagger)^H=\frac{1}{\sigma^2}\mathbf{K_s}\mathbf{A}^H\mathbf{K}_\mathbf{x}^{-1}\mathbf{A}\mathbf{K_s} \qquad (24)$$

and therefore

$$\begin{aligned}\mathbf{J}_{ij} &= \frac{2\mathbf{S}N}{\sigma^2}\mathrm{Re}\left\{\mathrm{tr}\left\{\mathbf{A}_i^H\mathbf{\Pi}^c\mathbf{A}_j\mathbf{K_s}\mathbf{A}^H\mathbf{K}_\mathbf{x}^{-1}\mathbf{A}\mathbf{K_s}\right\}\right\} \quad (25)\\ &= \frac{2\mathbf{S}N}{\sigma^2}\mathrm{Re}\left\{\left(\mathbf{D}^H\mathbf{\Pi}^c\mathbf{D}\right)_{ij}\left(\mathbf{I}\otimes\mathbf{K_s}\mathbf{A}^H\mathbf{K_x}\mathbf{A}\mathbf{K_s}\right)_{ij}\right\}\end{aligned}$$

Finally, the CRB matrix is found to be

$$CRB=\frac{\sigma^2}{2\mathbf{S}N}\mathrm{Re}\left\{\mathbf{D}^H\mathbf{\Pi}^c\mathbf{D}\odot\left(\mathbf{I}\otimes\mathbf{K_s}\mathbf{A}^H\mathbf{K_x}\mathbf{A}\mathbf{K_s}\right)^T\right\}^{-1} \qquad (26)$$

where $\odot$ denotes the element-wise matrix products.

3. SIMULATIONS

To demonstrate the the effectiveness and applicability of the proposed method, we consider the following scenario. A Uniform linear array of $M = 7$ sensors with inter-element spacing $\Delta = \frac{\lambda}{4}$ was used to estimate the locations of two sources located at $\{\theta_1, r_1\} = \{-5^0, 1.4\lambda\}$ and $\{\theta_2, r_2\} = \{20^0, 3\lambda\}$. The number of the snapshots (N) set to 200 and the SNR was varied from 0 to $20dB$. The proposed method was tested for $K = 1000$ independent trials. The resulting RMS errors of the estimated DOAs (in degrees) are shown in Figure 1 and Figure 2, while the corresponding RMS errors of the estimated ranges (in units of the wavelength) are shown in Figure 3 and Figure 4. The results were compared with the Cramer-Rao Bounds.

Based on the simulation results we made the following observations: -For high SNRs the $RMSE$s obtained from simulations becomes almost identical to the CRB results derived by modifying the results in [9].

4. CONCLUSIONS

In this paper, we derived CRBs for direction of arrival and range estimation of near-Field sources.

The Cramér-Rao bound for the near-field location estimators is provided, obey with the Monte Carlo simulations showing the performance of the algorithm based on Unitary Esprit and unconditional ML.

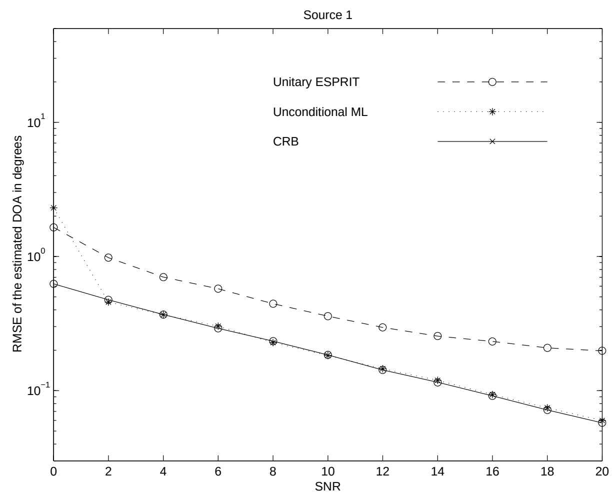

Figure 1: RMS error of the estimated DOA of source 1

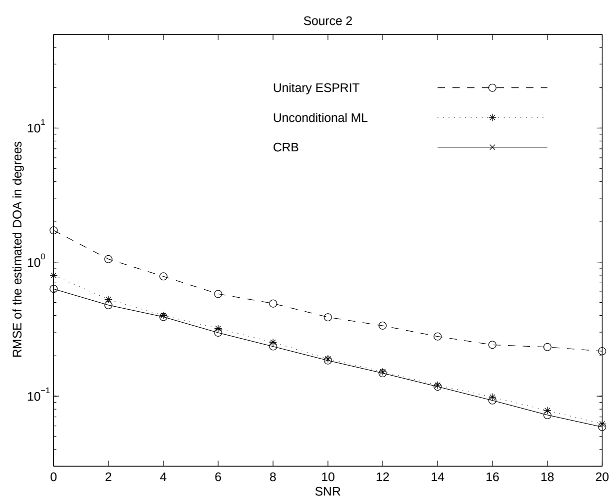

Figure 2: RMS error of the estimated DOA of source 2

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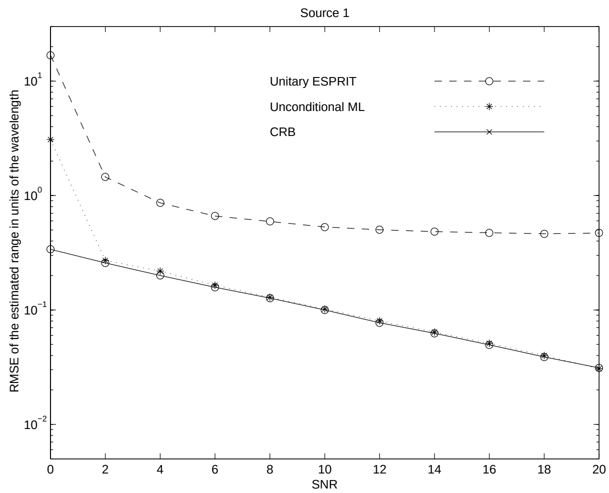

Figure 3: RMS error of the estimated range of source 1

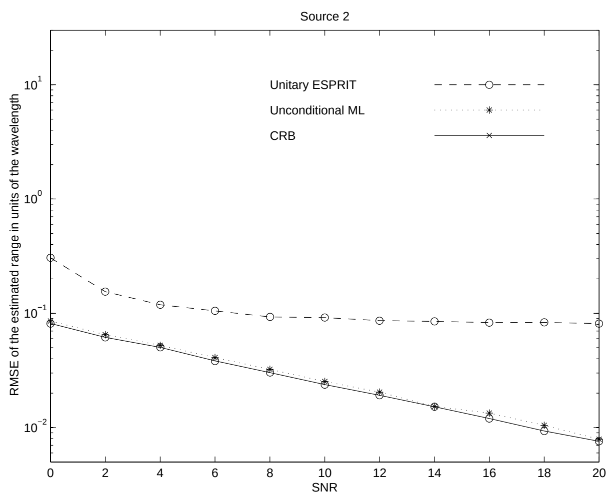

Figure 4: RMS error of the estimated range of source 2

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