A DECOMPOSITION ALGORITHM FOR MULTI-UNIVERSE FUZZY RELATIONS

Turhan Çiftçibaşı, Derya Altunay
Electrical and Electronics Eng. Dept.
Hacettepe University, Beytepe Ankara/Turkey

Abstract: Necessary and sufficient conditions for decomposition of multi-universe fuzzy truth functions in terms of one-universe truth functions are presented. An algorithm for decomposition is presented.

Keywords: Logic, proposition, decomposition.

I. INTRODUCTION

This paper is presented in "multi-universe fuzzy propositional logic" approach developed in [29]. This notation is outlined in Sections II through V with some new results. Different forms of fuzzy propositions are given with upper and lower limits. Some lemmas about distributivity of projection/shadow operations with respect to AND/OR operations are given. In Section VI, decomposition of a fuzzy relation as combination of propositions in subuniverses is discussed. Given two fuzzy propositions A and B in different universes, it is always possible to construct a fuzzy relation R in the common universe through a prescribed combination. However, the converse is not so obvious, if possible at all. In other words, given a fuzzy relation R, how would we know if it really represents a certain relationship between two fuzzy propositions A and B? Here the question is whether it is possible to find fuzzy propositions that conform to the given fuzzy relation, and if so, how to find them.

II. FUZZY PROPOSITION AND LOGICAL OPERATIONS

Definition [29]: Consider a set $U=\{u_i\}$ of "statements about a subject matter" that will be called the "universe set". A *fuzzy proposition* is defined in U, by a "truth function", which assigns a "truth value" in [0,1] for each element of U (for each statement).

The truth function is shown as $\mu_A(u)$ where A is the "name of the proposition", and the letter "u" is used as a generic element of the universe U (hence is a statement about the subject matter).

Definition: Given any two fuzzy propositions A, B in the same universe (statement set) U by their truth functions $\mu_A(u)$ and $\mu_B(u)$, the fuzzy propositions

"A AND B" shown as $(A \wedge B)$ or as $(A \cdot B)$ is defined as:

$\mu_{A \wedge B}(u) = \min \{\mu_A(u), \mu_B(u)\}$

"A OR B" shown as $(A \vee B)$ or as $(A+B)$ is defined as:

$\mu_{A \vee B}(u) = \max \{\mu_A(u), \mu_B(u)\}$

"NOT A" shown as $(\neg A)$ is defined as:

$\mu_{\neg A}(u) = 1 - \mu_A(u)$

III. SPECIAL FUZZY PROPOSITIONS AND THEIR INTERPRETATIONS

A proposition that has all of its truth values ≥ 0.5 is said to be a "semi-true" proposition.

A proposition that has all of its truth values ≤ 0.5 is said to be a "semi-false" proposition.

The truth functions of "semi-true" and "semi-false" propositions are said to be "large truth function" and "small truth function", respectively.

Example 3.1 [29]:

Let two different universes U and V be defined as below:

Subject matter of U = temperature of the process

$U = \{u_1, u_2, u_3, u_4\}$

= {low, below normal, above normal, high}

The universe has four elements, each of which is a statement (predicate) about the subject matter.

u_1 = temperature is low.

u_2 = temperature is below normal.

u_3 = temperature is above normal.

u_4 = temperature is high.

Subject matter of V = pressure of the process

$V=\{v_1, v_2, v_3, v_4, v_5\}$

={low, below medium, medium, above medium, high}

Let the fuzzy propositions be defined as:

$A_1 = \{1\ 0.6\ 0.4\ 0\}$ in U

$A_2 = \{0\ 0.6\ 1\ 0.6\}$ in U

$A_3 = \{0\ 0\ 0\ 1\}$ in U

$B_1 = \{0\ 0.2\ 0.6\ 0.9\ 1\}$ in V

$B_2 = \{0\ 0.6\ 1\ 0.6\ 0\}$ in V

Verbal interpretations (linguistic values) for the above propositions can be made as:

A_1= temperature cold

B_1= pressure high

A_2= temperature warm

B_2= pressure medium

A_3=temperature definitely hot

Then we have:

$\neg A_1$ (= temp not cold) = { 0 0.4 0.6 1 }

$\neg B_1$ (= pressure not high) = { 1 0.8 0.4 0.1 0 }

$\neg A_1 \wedge A_2$ (= temp not cold and warm)={0 0.4 0.6 0.6}

$\neg A_1 \vee A_2$ (=temp not cold or warm) = {0 0.6 1 1}

Note that

$A_1 \wedge \neg A_1$ (= temp cold and not cold)
= {0 0.4 0.4 0} (a semi-false proposition)

$B_2 \vee \neg B_2$ (= pressure medium or not medium)
={1 0.6 1 0.6 1} (a semi-true proposition)

Definition: A proposition, which has a constant truth value for all elements (has constant truth function) is said to be a *constant proposition.*

If the constant value of the truth function is α, this proposition is represented in short as $(\alpha)_U$ or as (α) when the universe is obvious. The constant proposition $(0)_U$ (having truth values all zero) is called the "null (empty) proposition" and the constant proposition $(1)_U$ (having truth values all unity) is called the "unity proposition". Interpretation (linguistic value) of the unity proposition (1) can be stated as "any", "all", "all of them true", "anyone is possible", and "anyone is accepted". Interpretation (linguistic value) of the null proposition (0) can be stated as "none", "none of them true", "no one is possible", and "no one is accepted".

Definition: A proposition (truth function) that gets the value "1" for only one statement (predicate) say u_i, and the value "0" for all other statements is said to be a *crisp proposition* and shown as $\hat{u}_i$. As an example, for the universe $U = \{u_1\ \ u_2\ \ u_3\ \ ...\ \ u_n\}$, the crisp proposition {0 1 0 ... 0} is shown as $\hat{u}_2$.

Definition: Given a proposition A in the universe U defined by the truth function $\mu_A(u)$,

$$\sigma_A = \max(A \cdot \neg A) = 1 - \min(A + \neg A) \tag{3.1}$$

is called the *degree of fuzziness* of A. Note that the two expressions on the right hand side of Eq.(3.1) are equal by De Morgan's rule.

Note that the degree of fuzziness can take a value between 0 and 0.5, and it is a measure of how far it is from crisp proposition. The degree of fuzziness of a crisp proposition is zero. If the degree of fuzziness is closer to 0.5, it is said to be farther away from being crisp.

The following lemmas regarding the fuzzy propositions, are important in the sequel.

Lemma 3.1: AND operation of a fuzzy proposition "A" via a constant proposition (α) limits (clips) the proposition A from above at value α.
Proof: Obvious.

Lemma 3.2: OR operation of a fuzzy proposition "A" via a constant proposition (α) limits (clips) the proposition A from below at value α.
Proof: Obvious.

IV. VARIOUS FORMS OF FUZZY PROPOSITIONS

Definition: The combination of fuzzy propositions via operations (AND, OR, NEGATION) are called *fuzzy propositional expressions.* $X = A \wedge (B \vee C)$, $Y = A \wedge \neg C \vee (D \wedge \neg A)$ are simple examples of fuzzy propositional expressions.

Note that during conversion and simplification of fuzzy expressions to different forms, commutative and distributive laws are applicable with exact equality for fuzzy case as well. Besides, the well known absorption rule

$$A + A \cdot B = A \tag{4.1}$$

is also applicable as exact equality for fuzzy case as well. However, the following equalities that are valid as exact equality for non-fuzzy case, are only approximate for fuzzy case:

$$A + \neg A \cdot B \ \cong\!\leq\ A + B \tag{4.2}$$

$$X \cdot (A + \neg A) \cong\!\leq X \tag{4.3}$$

$$X + (A \cdot \neg A) \cong\!\geq X \tag{4.4}$$

The signs "$\cong\!\leq$" and "$\cong\!\geq$" are used for meaning "truth function is smaller or equal, and also approximate" and "truth function greater or equal and also approximate", respectively. This sign becomes equality for non-fuzzy case.

As well known for non-fuzzy case, a propositional expression can be expressed in different forms [13], two of which are important:
1. Full Disjunctive Normal Form (FDNF) "OR combination of all AND terms".
2. Full Conjunctive Normal Form (FCNF) "AND combination of all OR terms".

Note that the different forms of propositional expressions are equal for non-fuzzy case, while are not equal for fuzzy case, but only approximate [13].

As an example note that the expression

$$X = A \cdot B + \neg A \cdot C + \neg A \cdot \neg B \tag{4.5}$$

can be expressed in Full Disjunctive Normal Form in three variables as

$$\begin{aligned} X &\cong\!\geq A \cdot B \cdot (C + \neg C) + \neg A \cdot C \cdot (B + \neg B) + \neg A \cdot \neg B \cdot (C + \neg C) \\ &= A \cdot B \cdot C + A \cdot B \cdot \neg C + \neg A \cdot C \cdot B + \neg A \cdot C \cdot \neg B \\ &\quad + \neg A \cdot \neg B \cdot C + \neg A \cdot \neg B \cdot \neg C = X_{FDNF} \end{aligned} \tag{4.6}$$

Right hand side (FDNF) is obviously smaller since each terms of X are combined with some terms by AND (minimum) operation. The same term can be expressed in Full Conjunctive Normal Form as

$$X_{FCNF} = (A + \neg B + C) \cdot (\neg A + B + C) \cdot (\neg A + B + \neg C) \tag{4.7}$$

Again, note the technique for converting FCNF to simplified form:

$$\begin{aligned} X_{FCNF} &= (A + \neg B + C) \cdot (\neg A + B + C) \cdot (\neg A + B + \neg C) \\ &= (A \cdot B + \neg A \cdot C + \neg A \cdot \neg B) + (A \cdot \neg A + B \cdot \neg B + C \cdot \neg C) \\ &= X + \text{semi-false term (small truth function)} >\cong X \end{aligned}$$

Lemma 4.1: All expressions obtained from each other by the above rules have FDNF and FCNF as upper and lower limits: $X_{FDNF} \cong\!\leq X \ \cong\!\leq X_{FCNF}$.
Proof: Obvious from equations (4.1) to (4.4) and the conversion rules to / from FDNF and FCNF.

FDNF and FCNF in three variables of the propositions (1) and (0) can be obtained as

$$(1)_{FDNF} = A \cdot B \cdot C + A \cdot B \cdot \neg C + A \cdot \neg B \cdot C + A \cdot \neg B \cdot \neg C$$
$$+ \neg A \cdot B \cdot C + \neg A \cdot B \cdot \neg C + \neg A \cdot \neg B \cdot C + \neg A \cdot \neg B \cdot \neg C$$

$$(0)_{FDNF} = (0)$$

$$(0)_{FCNF} = (A + B + C) \cdot (A + B + \neg C) \cdot (A + \neg B + C)$$
$$\cdot (A + \neg B + \neg C) \cdot (\neg A + B + C) \cdot (\neg A + B + \neg C)$$
$$\cdot (\neg A + \neg B + C) \cdot (\neg A + \neg B + \neg C)$$

$$(1)_{FCNF} = (1)$$

Note that

$(0.5) \leq (1)_{FDNF} \cong (1)$, i.e. $(1)_{FDNF}$ is a semi-true proposition close to (1)

$(0.5) \geq (0)_{FCNF} \cong (0)$, i.e. $(0)_{FCNF}$ is a semi-false proposition close to (0)

V. FUZZY OPERATIONS WITH MULTI-UNIVERSE REPRESENTATIONS

5.1 MULTI-UNIVERSE FUZZY RELATION

Definition: Let U and V be two universes (sets of statements about their subject matters). The *"two-universe fuzzy proposition"* (fuzzy relation) is defined in the universe U×V by assigning a truth value for each pair of statements (Zadeh [3]).

The truth function of a two-universe fuzzy proposition is represented as $\mu_R(u,v)$.

A finite, discrete two-universe fuzzy proposition can be shown in matrix form as:

$$A = \begin{vmatrix} 0.2 & 0.3 & 1 & 0.4 \\ 0.1 & 0.6 & 0 & 0.3 \\ 0.7 & 0 & 0.9 & 0 \end{vmatrix}$$

where the (i,j)'th element shows the truth value of the (i,j)'th statement pair.

Multi-universe fuzzy relation is similarly defined as the fuzzy proposition in the universe $U_1 \times U_2 \times .. \times U_n$ by the truth function $\mu_R(u_1,u_2,..,u_n)$.

5.2 REPRESENTATION OF A FUZZY PROPOSITION IN A HIGHER UNIVERSE

Definition: Consider a fuzzy proposition A defined in the universe U by the truth function $\mu_A(u)$. The fuzzy proposition defined in the Cartesian product universe U×V by the truth function $\mu_{EXT\{A\}}(u,v) = \mu_A(u)$ is called the "Extension of the fuzzy proposition A into the higher universe U×V" or the "Representation of the fuzzy proposition in the higher universe U×V" and is shown in short as $\{A\}_{U\times V}$, as $Ext_{U\times V}\{A\}_U$ or as $Ext\{A\}$ when the universes are obvious.

In matrix representation (of finite, discrete case), the known values of the column/row of the fuzzy proposition are copied as the values of the newly added dimension.

Example 5.1:
For the universes
$U = \{ u_1, u_2, u_3, u_4\}$, $V = \{ v_1, v_2, v_3, v_4, v_5\}$
let the fuzzy propositions defined as:
$A_1 = \{1 \ 0.6 \ 0.4 \ 0\}$ in U
$B_1 = \{0 \ 0.2 \ 0.6 \ 0.9 \ 1\}$ in V
Then we obtain the extensions as:

$$Ext_{U\times V}\{A_1\} = \{A_1\}_{U\times V} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.6 & 0.6 & 0.6 & 0.6 & 0.6 \\ 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$Ext_{U\times V}\{B_1\} = \{B_1\}_{U\times V} = \begin{vmatrix} 0 & 0.2 & 0.6 & 0.9 & 1 \\ 0 & 0.2 & 0.6 & 0.9 & 1 \\ 0 & 0.2 & 0.6 & 0.9 & 1 \\ 0 & 0.2 & 0.6 & 0.9 & 1 \end{vmatrix}$$

5.3 PROJECTION AND SHADOW OPERATIONS ON A FUZZY PROPOSITION

Assume that a two-universe proposition R defined by the truth function $\mu_R(u,v)$ is given, but we only consider one of the universes as important, say the universe U. By "projection" and "shadow" operations we obtain one-universe propositions from two-universe proposition R.

Definition: Given a fuzzy proposition "R" on U×V defined by the truth function $\mu_R(u,v)$, the fuzzy proposition "A" defined on U by the truth function

$$\mu_A(u) = \max_{v \in V} \{\mu_R(u,v)\}$$

is called the *projection* of R on U (Zadeh [5]), and also shown as $A = Proj_U\{R\}_{U\times V}$.

Definition: Given a fuzzy proposition "R" on U×V defined by the truth function $\mu_R(u,v)$, the fuzzy proposition "A" defined on U by the truth function

$$\mu_A(u) = \min_{v \in V} \{\mu_R(u,v)\}$$

is called the *shadow* of R on U and also shown as $A = Shad_U\{R\}_{U\times V}$.

Note that the projection operation gives emphasis on higher truth values, while the shadow operation gives emphasis on lower truth values. In other words, by projection operation information are collected about "what is true" and lower truth values are disregarded. The result of the shadow operation gives information about "what is false" and the higher truth values are disregarded. We believe that both are equally important, hence the two results should be considered together.

5.4 LOGICAL OPERATIONS ON TWO FUZZY PROPOSITIONS IN DIFFERENT UNIVERSES

"A and B" ($A \wedge B$ or shortly $A \cdot B$) of the fuzzy proposition A (defined by $\mu_A(u)$ in the universe U) and the fuzzy proposition B (defined by $\mu_B(v)$ in the universe V) is defined by AND operation in their extended universe representation as:

$$\{A\}_U \cdot \{B\}_V = \{A\}_{U\times V} \cdot \{B\}_{U\times V}$$

Note that this is equivalent to

$$\mu_{A\cdot B}(u,v) = \min\{\mu_A(u), \mu_B(v)\}$$

which is the same as "cross product" operation defined by Zadeh. (Zadeh and Mamdani, also interpreted the cross product operation as AND operation).

"OR" operation of two fuzzy propositions in different universes is similarly defined by extending to the higher common universe, and performing the operation in this universe.

Example 5.2:
Consider the same universes in Example 5.1, with the fuzzy propositions:

$A_1 = \{1\ \ 0.6\ \ 0\ \ 0\}$
$A_2 = \{0\ \ 0.2\ \ 0.8\ \ 1\}$
$B_1 = \{1\ \ 0.8\ \ 0.1\ \ 0\ \ 0\}$
$B_2 = \{0\ \ 0\ \ 0.1\ \ 0.8\ \ 1\}$

Then, from the definition of "AND" operation

$$A_2 \wedge B_1 = \{0\ \ 0.2\ \ 0.8\ \ 1\}_U \wedge \{1\ \ 0.8\ \ 0.1\ \ 0\ \ 0\}_V$$

$$= \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} \wedge \begin{vmatrix} 1 & 0.8 & 0.1 & 0 & 0 \\ 1 & 0.8 & 0.1 & 0 & 0 \\ 1 & 0.8 & 0.1 & 0 & 0 \\ 1 & 0.8 & 0.1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.1 & 0 & 0 \\ 0.8 & 0.8 & 0.1 & 0 & 0 \\ 1 & 0.8 & 0.1 & 0 & 0 \end{vmatrix}$$

Lemma 5.1: [29] For any fuzzy propositions A_1, A_2 in the same universe U,

$Ext_{U\times V}\{A_1 \cdot A_2\}_U = Ext_{U\times V}\{A_1\}_U \cdot Ext_{U\times V}\{A_2\}_U$
$Ext_{U\times V}\{A_1 + A_2\}_U = Ext_{U\times V}\{A_1\}_U + Ext_{U\times V}\{A_2\}_U$
$Ext_{U\times V}\{\neg A_1\}_U = \neg\, Ext_{U\times V}\{A_1\}_U$

In words, "performing Boolean operations and then taking extension into the higher universe" is equivalent to "first taking extension and then performing the Boolean operation in the extended universe".
Proof: Obvious by definition.

Extension of a fuzzy proposition into a higher universe will have the same physical interpretation with the original proposition. In fact, in Example 5.1, $Ext_{U\times V}\{A_1\}$ can be interpreted as "temperature is cold, pressure is any" hence has been also shown as $\{A_1\}_{U\times V}$.

Equivalence axioms can be proved between a fuzzy proposition in the universe U and its extensions in the higher universes. It should be noted that since the "projection" and "shadow" of the "extension fuzzy proposition" into the "original lower universe" is equal to the original proposition, projection or shadow operation can be accepted as the inverse procedure of the extension operation.

Both one-universe and two-universe models can represent the same system equivalently. Even after the one-universe model has been chosen and calculations (by using AND, OR, NOT operations and their combinations) have been made on this system, whenever this model becomes insufficient, one can just add a new universe and continue. Since all calculations that were made up to that time are valid in the higher universe, after this step calculations can continue in the higher universe.

The representation of a proposition "A" after the extension into the higher universe U×V will also be called with the same name "A" assuming that all definitions and calculations that were made up to that time were made in this higher universe. The original proposition in the original universe U will then be represented as $\{A\}_U$. If representations, of both one-universe, and two-universe fuzzy propositions should be shown in one equation, as in *"extension", projection", "shadow"* operations, the following notation will be used [29]:

$Ext_{U\times V}\{A\}_U$ extension into U×V of the fuzzy proposition A in U.
$Proj_U\{A\}_{U\times V}$ projection into U of the fuzzy proposition A in U×V.
$Shad_U\{A\}_{U\times V}$ shadow into U of the fuzzy proposition A in U×V.

5.5 LOGIC OF COMPOSITION OPERATION

Definition: Let U=(u_1 u_2 u_3 ... u_n) and V=(v_1 v_2 v_3 ... v_m) be two universes. Consider the fuzzy relation (two-universe fuzzy proposition) R given in U×V in matrix form, and a fuzzy proposition A in U given in vector form. *Composition* of R and A is defined by Zadeh [13-16] as:

$$R \circ A = \begin{vmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & & r_{2n} \\ \dots & & & \\ r_{m1} & r_{m2} & & r_{mn} \end{vmatrix} \circ \begin{vmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{vmatrix}$$

$$= \begin{vmatrix} \max\{(r_{11}\wedge a_1), (r_{12}\wedge a_2), \dots, (r_{1n}\wedge a_n)\} \\ \max\{(r_{21}\wedge a_1), (r_{22}\wedge a_2), \dots, (r_{2n}\wedge a_n)\} \\ \dots \\ \max\{(r_{m1}\wedge a_1), (r_{m2}\wedge a_2), \dots, (r_{mn}\wedge a_n)\} \end{vmatrix}$$

The result is an *m-vector* (proposition in the universe V). Due to similarity of this operation to matrix multiplication, this operation is called as "max-min" matrix operation.

As Zadeh also stated, this operation is equivalent to "AND operation followed by projection operation". For this reason we prefer to use the representation $Proj_V\{R\wedge A\}_{U\times V}$ instead of the representation $R\circ A$. This is more meaningful, because it explicitly states the logic of composition operation as *"AND operation in the higher universe and projection into the requested universe."*

5.6 CONTRACTION AFTER EXTENSION

As stated above, if a proposition is first extended to a higher universe, and then projection/ shadow is taken back into the original universe, the original proposition is obtained. Hence:

$$Proj_U\{\, Ext_{U\times V}\{A\}_U \,\} = A \qquad (5.1)$$
$$Shad_U\{\, Ext_{U\times V}\{A\}_U \,\} = A \qquad (5.2)$$

What is obtained if the proposition A in the universe U is extended into the higher universe U×V and projection/shadow is taken into the other universe V? The result is given below [29]:

Lemma 5.2:

a) $Proj_V\{Ext_{U\times V}\{A\}_U\} = (\alpha_A)_V$ (5.3)

b) $Shad_V\{Ext_{U\times V}\{A\}_U\} = (\varepsilon_A)_V$ (5.4)

where α_A,ε_A are the maximum and the minimum values of the proposition A, and $(\alpha_A)_V,(\varepsilon_A)_V$ are constant propositions in the universe V having values α_A, ε_A, respectively.

Proof: Consider that the proposition A in U has been extended into U×V by copying the proposition A to all columns. If the maximum element (say the k'th element) of A is α_A, then the k'th row of the matrix is all α_A, while the other columns are smaller. Projection into V gives the proposition $(\alpha_A)_V$. Part-b is proved similarly.

Equations (5.1) and (5.2) are called "returning back into the original universe", while (5.3) and (5.4) can be called as "indirect projection (indirect shadow) from a one-universe into another one-universe".

Cartesian product (AND operation) of two fuzzy propositions A,B in different universes U,V gives the result in the universe U×V. Is it possible to obtain the original propositions A , B by taking the projection back into the original universes? The reply is given in the following lemmas [29]:

Lemma 5.3: Consider the fuzzy propositions A and B in the universes U and V respectively. If the maximum truth value of the proposition B is less than "1", say α_B, then the projection $Proj_U\{A\cdot B\}_{U\times V}$ is different from the proposition A such that the truth values above α_B are decreased to the value α_B (clipped from above by α_B). The counterpart is also applicable.

Proof: Let the maximum element (say the k'th element) of B is α_B. In the extended universe representation of B into the universe U×V (copying the row vector B into all rows) the k'th column of the extended matrix is all α_B (this column is constant fuzzy proposition $(\alpha_B)_U$), while the other columns are smaller. Extension of the fuzzy proposition A into U×V makes all columns copies of the column vector A. AND operation of these two extension matrices makes the k'th column $A\cdot(\alpha_B)_U$ (A limited from above by α_B) while the other columns are smaller. Projection into U gives $A\cdot(\alpha_B)_U$. Hence the result.

Lemma 5.4: Consider the fuzzy propositions A and B in the universes U and V respectively. If the minimum truth value of the proposition B is greater than "0", say ε_B, then the shadow $Shad_U\{A+B\}_{U\times V}$ is different from the proposition A such that the truth values below ε_B are increased to the value ε_B (clipped from below by ε_B). The counterpart is also applicable.

Proof : Can be made similar to the proof of Lemma 5.3.

5.7 DISTRIBUTIVITY PROPERTIES OF PROJECTION / SHADOW OPERATIONS

Consider projection / shadow operation into a one-universe of a multi-universe propositional expression as:

$Proj_U\{A+A\cdot\neg B+\neg A\cdot B\}$

$Shad_U\{A\cdot B+\neg B+\neg A\}$

where A,B are propositions in the universe U,V respectively. Is it possible to use distributive law so that projections and shadows can be taken separately? The answer is given in the following lemmas:

Lemma 5.5: Projection operation is distributive with respect to OR operation.

Lemma 5.6: Shadow operation is distributive with respect to AND operation.

Proofs of the above lemmas are obvious since both projection and OR operations are performed by maximum operations, and since both shadow and AND operations are performed by minimum operations.

Projection operation is in general not distributive with respect to AND operation. Similarly, shadow operation is in general not distributive with respect to OR operation. However, the following lemmas give important sufficient conditions for distributivity.

Lemma 5.7: [29] If the AND operation is between propositions which are extensions of separate, originally one-universe propositions, then projection operation is distributive with respect to AND operation.

$Proj_U\{A\cdot B\}_{U\times V}$

$= Proj_U\{A\}_{U\times V}\cdot Proj_U\{B\}_{U\times V} = A\cdot(\alpha_B)_U$

Proof of Lemma 5.3 above is also the proof of this lemma.

Lemma 5.8: [29] If the OR operation is between propositions which are extensions of separate, originally one-universe propositions, then shadow operation is distributive with respect to OR operation. Proof can be made as in Lemmas 5.3 and 5.4.

Let A, B be fuzzy propositions in the universes U,V respectively. Combining these propositions via AND/OR operations gives the results in the higher universe U×V. What is obtained if we take projection/shadow back into the original universes? The replies are given in Lemma 5.9:

Lemma 5.9:

$Proj_U\{A\cdot B\}_{U\times V}$

$= Proj_U\{A\}_{U\times V}\cdot Proj_U\{B\}_{U\times V} = A\cdot(\alpha_B)_U$ (5.5)

= A in U limited from above by the maximum value of B.

$Proj_V\{A\cdot B\}_{U\times V}$

$= Proj_V\{A\}_{U\times V}\cdot Proj_V\{B\}_{U\times V} = (\alpha_A)_V\cdot B$ (5.6)

= B in V limited from above by the maximum value of A.

$Proj_U \{A+B\}_{U\times V}$
$= Proj_U \{A\}_{U\times V} + Proj_U \{B\}_{U\times V} = A + (\alpha_B)_U$ (5.7)
= A in U limited from below by the maximum value of B.

$Proj_V \{A+B\}_{U\times V}$
$= Proj_V\{A\}_{U\times V} + Proj_V \{B\}_{U\times V} = (\alpha_A)_V + B$ (5.8)
= B in V limited from below by the maximum value of A.

$Shad_U \{A\cdot B\}_{U\times V}$
$= Shad_U \{A\}_{U\times V} \cdot Shad_U \{B\}_{U\times V} = A \cdot (\varepsilon_B)_U$ (5.9)
= A in U limited from above by the minimum value of B.

$Shad_V \{A\cdot B\}_{U\times V}$
$= Shad_V \{A\}_{U\times V} \cdot Shad_V \{B\}_{U\times V} = (\varepsilon_A)_V \cdot B$ (5.10)
= B in V limited from above by the minimum value of A.

$Shad_U \{A+B\}_{U\times V}$
$= Shad_U\{A\}_{U\times V} + Shad_U\{B\}_{U\times V} = A+(\varepsilon_B)_U$ (5.11)
= A in U limited from below by the minimum value of B.

$Shad_V \{A+B\}_{U\times V}$
$= Shad_V\{A\}_{U\times V} + Shad_V\{B\}_{U\times V} = (\varepsilon_A)_V+B$ (5.12)
= B in V limited from below by the minimum value of A.

The constants α_A, α_B are the maximum values of the propositions A, B respectively, and the constants ε_A, ε_B are the minimum values of the propositions A, B respectively. These results give simpler proofs and extensions of Lemmas 5.3 and 5.4.

VI. DECOMPOSITION OF A MULTI-UNIVERSE PROPOSITION INTO ONE-UNIVERSE PROPOSITIONS

Given a two-universe fuzzy proposition (fuzzy relation) R in the universe U×V, is it possible to find propositions A and B in the universes U and V satisfying $R=A\wedge B$ or $R=A\vee B$? This property is called as *"decomposition property"* and discussed in [33-36] in fuzzy set-theoretic approach. The decomposition property will be discussed with multi-universe fuzzy propositional logic approach, necessary and sufficient conditions will be given, and an algorithm for decomposition will be presented.

Lemma 6.1:

a) A necessary and sufficient condition for the relation R to be expressed as $R=A\wedge B$ is that there will be a complete row or column consisting of the minimum element of R, and the remaining matrix R' obtained by deleting this row or column should have the same property ($R' = A' \wedge B'$ is satisfied for some A' and B').

b) A necessary and sufficient condition for the relation R to be expressed as $R=A\vee B$ is that there will be a complete row or column consisting of the maximum element of R, and the remaining matrix R' obtained by deleting this row or column should have the same property ($R' = A' \vee B'$ is satisfied for some A' and B').

Proof of (a): Necessity of one full row with minimum value: Let R be decomposable, i.e. for some A, B in U and V respectively, $R=\{A\}_U\wedge\{B\}_V$ is satisfied and let the minimum element of both A and B be in the k'th element of A having value ε. Extension of A into U×V will give copies of A to the other columns, hence the k'th row will be all ε. Since ε is the minimum value of both A and B, AND operation via extension of B makes the k'th row all ε, while all other elements greater or equal.

Necessity of decomposability of the remaining matrix: Without loss of generality assume that the minimum value row is the last row of R. Then the following operations in partitioned form is applicable:

$$R = \begin{vmatrix} R' \\ (\varepsilon) \end{vmatrix} = A \wedge B = \begin{vmatrix} A' \\ A'' \end{vmatrix} \wedge B$$

$$= \begin{vmatrix} A'\wedge B \\ A''\wedge B \end{vmatrix} = \begin{vmatrix} A'\wedge B \\ (\varepsilon) \end{vmatrix}$$

Hence R' should also be decomposable as $A'\wedge B$.

Sufficiency: Without loss of generality assume that the minimum element line is the last row of R with values all ε, and let R' be the matrix obtained by disregarding the k'th row of R, satisfying $R'=A'\wedge B'$ for some A' and B'. From the above partitioned matrix operation, it is obvious that R is also decomposable.

Proof of (b): Can be obtained similarly.

The above lemmas give us a very easy algorithm for obtaining decompositions. Note that by disregarding a constant row (column), an element of A (B) is found. In the remaining matrix, a column or row should be searched. If the row or column with minimum (maximum) element is not present at any stage, then decomposition as $R=A\wedge B$ ($R=A\vee B$) is not possible.

Example 6.1: Decomposition Algorithm for $R=A\cdot B$:

						A	step no
	0.9	0.6	0.9	0.3	0.8	0.9	6
	0.1	0.1	0.1	0.1	0.1	0.1	1
	0.95	0.6	0.95	0.3	0.8	0.95	7
	0.7	0.6	0.7	0.3	0.7	0.7	4
B	*x*	0.6	*x*	0.3	0.8		
		3		2	5	step no	

Step-1: The matrix R has the minimum element of 0.1 in the second row. Hence the second element of A is 0.1.

Step-2: Disregarding the second row of R, the minimum element is 0.3, in the fourth column. Hence, the fourth element of B is 0.3.

...

After step-7, the remaining two elements (shown as *x*) can have any value greater than or equal to "the last used value" 0.95.

The step numbers are given above for reference.

Example 6.2: Decomposition Algorithm for R=A+B:

						A	step no
	0	0.9	0.1	0.5	0.2	x	
	0.9	0.9	0.9	0.9	0.9	0.9	2
	0.3	0.9	0.3	0.5	0.3	0.3	4
	0	0.9	0.1	0.5	0.2	x	
B	0	0.9	0.1	0.5	0.2		
step no	7	1	6	3	5		

Step-1: The matrix R has the maximum element of 0.9 in the second column. Hence, the second element of B is 0.9.

Step-2: Disregarding the second column of R, the maximum element is 0.9, in the 2nd row. Hence, the second element of A is 0.9.

...

After step-7, the remaining two elements (shown as x) can have any value less than or equal to "the last used value" 0, (hence they should be zero).

Note that in Example 6.1 if the terms shown by "x" are selected as the minimum value (the last used value 0.95), $A = Proj_U\{R\}$ and $B = Proj_V\{R\}$ are obtained. Note again that in Example 6.2, if the terms shown by "x" are selected as the maximum value (last used value 0), $A=Shad_U\{R\}$ and $B=Shad_V\{R\}$ are obtained.

The lemmas given below state that these are in fact always valid [33]:

Lemma 6.2: If the relation R is decomposable as $R=A\cdot B$, then among all possible decompositions, $A_1=Proj_U\{R\}$ and $B_1=Proj_V\{R\}$ are the minimum propositions.

Proof: For any decomposition A and B satisfying $R=A\cdot B$, by Eq. (5.5) and (5.6),

$A_1=Proj_U\{R\}= Proj_U\{A\cdot B\}_{U\times V}$

$= Proj_U\{A\}_{U\times V} \cdot Proj_U\{B\}_{U\times V} = A \cdot (\alpha_B)_U$

which is $\leq A$.

$B_1=Proj_V\{R\}= Proj_V\{A\cdot B\}_{U\times V}$

$= Proj_V\{A\}_{U\times V} \cdot Proj_V\{B\}_{U\times V} = (\alpha_A)_V \cdot B$

which is $\leq B$.

and

$A_1\cdot B_1 = Proj_U\{R\}\cdot Proj_V\{R\}$

$= (A\cdot(\alpha_B)_U)\cdot(B\cdot(\alpha_A)_V) = (A\cdot(\alpha_A)_V)\cdot(B\cdot(\alpha_B)_U)$

= (A in U limited by the maximum value of A)$\cdot$ (B in V limited by the maximum value of B)

$= A\cdot B = R$

Therefore A_1 (projection in U) and B_1 (projection in V) are also decompositions and are smaller than or equal to any other possible decomposition.

Lemma 6.3: If the relation R is decomposable as R=A+B, then among all possible decompositions, $A_1=Shad_U\{R\}$ and $B_1=Shad_V\{R\}$ are the maximum propositions.

Proof: Can be made as in Lemma 6.2.

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