LOCAL UNIFORM EXPONENTIAL STABILIZATION OF A CLASS OF NONLINEAR NON-AUTONOMOUS SYSTEMS

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ABSTRACT

In this paper a novel local uniform exponential stabilization for a class of nonlinear systems is investigated. A continuous and time-varying feedback control is constructed under which the local uniform exponential convergence for such systems is guaranteed. It is seen that the convergence rate and convergence region are variable parameters and they are chosen by the designer. The proposed method dose not need to construct Lyapunov functions and a time-invariant one is introduced. Some illustrative examples are also provided to show the effectiveness of the proposed method.

I. INTRODUCTION

Given a control system, the first and most important question about its various properties is the stability [1]. The most useful and general approach for studying the stability of nonlinear control systems is the theory introduced in the late 19th century by the Russian mathematician Alexander Mikhailovich Lyapunov. Lyapunov's work includes two methods for stability analysis, indirect (linearization) method and direct (Lyapunov function) method, which determines the stability of equilibrium points. The indirect method draws conclusion about a nonlinear system's local stability close to an equilibrium point. The direct method is not restricted to be close to equilibrium and determines the stability by considering a scalar continuous and differentiable function for the system and examining the function time variation [1-5]. However it does not provide any coherent methodology for constructing such a function. The investigation of stability analysis of nonlinear systems using the direct Lyapunov method has produced a vast body of important results and has been widely studied.

The Lyapunov method that was introduced for analysis also is a useful tool for control design (synthesis) of the nonlinear systems. It is recognized that the Lyapunov function method serves as a main technique to reduce a given complicated system to a relatively simple system and provide useful applications to control theory [6]. The objective of control design for physical systems is to construct a feedback control law which the closed-loop system satisfies the desired behavior [3]. For linear time invariant (LTI) systems there are a wide variety of controller design techniques that achieve a range of performance objectives including state regulation, tracking desired trajectories, etc. The linear quadratic regulator, H-infinity and other robust control techniques, classical approaches such as root locus and Bode design techniques are different methodologies for achieving the desired behavior. But there are no universal techniques for nonlinear systems. Different nonlinear systems must generally be considered as a separate design problem. In many design methods based on Lyapunov's direct method a Lyapunov function is constructed to guarantee the convergence of trajectories to an equilibrium point or an equilibrium set. Among them are Lyapunov redesign, back stepping, sliding mode and adaptive control [1-5].

There have been a number of interesting developments in searching the stabilization law for nonlinear systems, but most have been restricted to find the asymptotic stabilization control law. Unlike linear systems, where the asymptotic stability implies exponential stability, the exponential stability for nonlinear systems in general may not be easily verified. Some recent investigations which have dealt with exponential stabilization for nonlinear systems can be seen for example [6-12]. Sufficient condition for stability of a class of nonlinear time-varying differential equations was proposed in [6]. In [7] exponential stabilization for single chained form systems by means of a static discontinuous feedback was investigated. An exponential stabilizing controller for an open-loop unstable bilinear system was proposed in [8]. In [9] an extended chain form is investigated and ultimate exponential stabilization is achieved. The problem of exponential stabilization of a drift less nonholonomic chained system is addressed and solved by means of time-varying control law in [10]. In [11] the problem of robust exponential stabilization of a class of uncertain dynamic systems with time varying delays and bounded controllers is proposed. Finally [12] proposed a global exponential stabilization for uncertain linear systems via output feedback.

In this paper two novel approaches based on Lyapunov method for local exponential stabilization of a class of non-autonomous nonlinear systems are introduced, until the systems requirement satisfied. The proposed method dose not need to construct Lyapunov functions and a time-invariant one is introduced. The paper is organized as follows: Section II introduces notation, definitions and other preliminaries. The main results and some illustrative examples are stated in section III. Finally section IV is conclusion.

II. PRELIMINARIES

Throughout this paper, R^n is the n-dimensional real space, $R_{\geq k}$ is the real numbers greater or equal to *k*, R^+ denotes the positive real numbers, $\|x\|$ is Euclidean norm of $x \in R^n$, $\nabla V(x)$ is Gradient of smooth scalar function $V(x)$, B_R denotes the spherical region (ball) defined by $\|x\| < R$, and $|a|$ is absolute value of real number a.

Definition 1: A function $V : D \to R$ is said to be positive definite in D if it satisfies the following conditions:

i. $0 \in D$ and $V(0) = 0$
ii. $V(x) > 0$ in $D - \{0\}$

Definition 2: Let $V : D \to R$ and $f : D \to R^n$. The Lie derivative of *V* along *f*, denoted by $L_f V$, is defined by

$$L_f V(x) = \left(\frac{\partial V}{\partial x}\right)^T f(x) = \nabla^T V . f(x)$$

Consider the nonlinear system described by the time-varying differential equations:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) \qquad t \geq 0 \\ x(t_0) &= x_0 \qquad t_0 \geq 0 \end{aligned} \tag{1}$$

where, $x(t) \in R^n$, $f(t,x) : R^+ \times R^n \to R^n$ is a given nonlinear function satisfying $f(t,0) = 0$ for all $t \in R^+$. It is assumed that the existence of system's (1) solution is guaranteed.

Definition 3: Consider, the function $V(x) : D \to R$ has a continuous partial derivatives, and is positive definite in a ball B_R, if the time derivative of $V(x)$ along any state trajectory of system (1) is negative definite i.e. $\dot{V}(x) < 0$, then $V(x)$ is said to be a Lyapunov function for system (1).

Definition 4: The equilibrium point *x*=0 of system (1) is locally exponentially stable if any solution of (1) satisfies

$$\|x(t, x_0)\| \leq \alpha(t_0, \|x_0\|) e^{-\lambda(t-t_0)} \qquad \forall t \geq t_0 \tag{2}$$

whenever, $\|x(t_0)\| < \delta$ and $\alpha(r,s) : R^+ \times R^+ \to R^+$ is a non-negative function increasing in $r \in R^+$, and λ is a positive constant. It is said to be locally uniformly exponentially stable if the function $\alpha(\cdot)$ dose not depend on t_0. It is said to be globally exponentially stable if (2) is satisfied for any $x \in R^n$.

According to system (1), consider a class of nonlinear non-autonomous control systems as

$$\dot{x}(t) = f(t, x(t), u(t)) \qquad , t \geq 0 \tag{3}$$

that $x \in R^n$, $u \in R^m$, $f(t,x,u) : R^+ \times R^n \times R^m \to R^n$.

Definition 5: Control system (3) is exponentially stabilizable by the time-varying state feedback $u = \varphi(t, x(t))$, where $\varphi(\cdot,\cdot) : R^+ \times R^n \to R^m$ if the closed-loop system

$$\dot{x}(t) = f(t, x(t), \varphi(t, x(t)))$$

is exponentially stable.

III. PROBLEM FORMULATION AND MAIN RESULTS

Lemma 1: Consider that the origin is an equilibrium state for system (1). If in a neighborhood *D* of the equilibrium state *x*=0, there exist a scalar differentiable function $V(\cdot) : D \to R$ such that

i. $V(x)$ is positive definite
ii. There exist positive constant α, β, p such that
$$\alpha \|x\|^p \leq V(x) < \beta$$
iii. The derivative of $V(\cdot)$ along any solution of (1) satisfies
$$\dot{V}(x) = \lambda V(x)(V(x) - \beta)$$

which λ is a positive constant.

Then system (1) is locally uniformly exponentially stable and its convergence rate is $\frac{\lambda\beta}{p}$.

Proof: Since V is positive definite and

$$\dot{V} = \lambda V(V-\beta)$$

and

$$\alpha\|x\|^p \le V < \beta$$

then, $\dot{V}$ is negative definite, clearly (1) is asymptotically stable. To show the exponential stability, we have

$$\frac{dV}{dt} = \lambda V(V-\beta)$$
$$= -\lambda V(\beta - V)$$

solving the above differential equation leads to

$$V(x) = \frac{k\,\beta\,e^{-\lambda\beta t}}{1+k\,e^{-\lambda\beta t}}$$

where

$$k = \frac{V(0)}{\beta - V(0)}$$

Because $V(0) < \beta$, so $k > 0$, and

$$V(x) < k\,\beta\,e^{-\lambda\beta t} \tag{4}$$

Furthermore,

$$\alpha\|x(t)\|^p \le V(x) \tag{5}$$

so, comparing (4) and (5) leads to

$$\|x(t)\| < \left[\frac{k\beta}{\alpha}\right]^{1/p} e^{-\frac{\lambda\beta t}{p}} \tag{6}$$

The inequality (6) shows that (1) is locally uniformly exponentially stable with convergence rate of $\frac{\lambda\beta}{p}$. □

Definition 6: Consider an $n\times n$ symmetric positive-definite matrix P and $y \in R^n$, then $E(y,P)$ is called a n-dimensional ellipsoid with center y if,

$$E(y,P) = \{x \in R^n : (x-y)^T P(x-y) \le 1\} \tag{7}$$

Now, consider a class of non-autonomous nonlinear SISO control systems in the form of

$$\dot{x} = f(t,x) + g(x)u \quad t \ge 0$$
$$x(0) = x_0 \tag{8}$$

where, $f(t,x): R^+ \times R^n \to R^n$ and $g(x): R^n \to R^n$, are differentiable for any $x(t) \in R^n$. Assume that the origin is an equilibrium point of the unforced system. It should be noted that this class of systems is very famous and many physical systems can be modeled as (8).

Consider, $V(x)$ as follows

$$V(x) = x^T Px \le R \tag{9}$$

where, P is a diagonal and positive-definite matrix and R, is a positive constant, also assume we set $\dot{V}$ as

$$\dot{V} = V(V-R) = h(x) \tag{10}$$

Clearly V is in the form of lemma 1. Now, consider V, be a Lyapunov function for (8). From (8) it is obtained,

$$\frac{dV}{dt} = \frac{dV}{dx}\dot{x}$$
$$= \nabla^T V.f(t,x) + \nabla^T V.g(x)u$$
$$= \nabla^T V.f(t,x) + L_g Vu \tag{11}$$

Equations (10) and (11) lead to,

$$V(V-R) = \nabla^T V.f(t,x) + L_g Vu \tag{12}$$

Equation (12) and lemma 1 show that the system defined in the form of (8) is locally exponentially stabilizable for any $x_0 \in E(0,\frac{P}{R})$ and under control $\varphi(t,x)$ if, $h(x) - \nabla^T V.f(t,x)$ divides $L_g V$, with $\varphi(t,x)$ be the divisor and a continuous function for any $x \in E(0,\frac{P}{R})$, i.e.

$$h(x) - \nabla^T V.f(t,x) = \varphi(t,x)L_g V$$
$$h(x) - \nabla^T V.f(t,x) \text{ divides } L_g V \tag{13}$$
$$\varphi(t,x) \text{ is continuous for any } x \in E(0,\frac{P}{R})$$

where, the division condition lets using limited control signal. It can be seen that the convergence rate and convergence region are variables and can be chosen by the designer. If $g(x)$ be constant this method can be applied to any first-order nonlinear systems in the form of

(8), because condition (13) is always satisfied. For higher-order systems checking (13) is necessary.
Now, to show the applicability of the method consider following examples.

Example 1: Consider a nonlinear first-order system as follows,

$$\dot{x} = -x^2 - 2x(1 + 2\cos(5t)) + u \qquad (14)$$

By Choosing $V(x) = x^2 < 1$ and from (10) we set $\dot{V}(x) = x^2(x^2 - 1) = h(x)$ it is clear that $h(x) - \nabla^T V.f(t,x)$ divides $L_g V = 2x$. Therefore, from lemma 1 and condition (13), system (14) is locally exponentially stabilizable for any $x_0 \in (-1,1)$ and under control

$$u = \frac{1}{2}(x^3 - x) + x^2 + 2x(1 + 2\cos(5t)) \qquad (15)$$

Figures 1(a) and (b) show the system response for $x_0 = 0.8$, and the control signal, respectively.

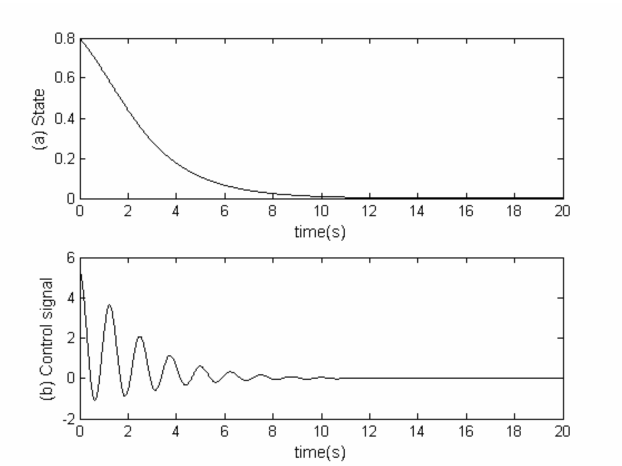

Figure 1. (a) Response of system (14) with $x_0 = 0.8$
(b) Control signal $u(t)$ for the system defined by (14)

Example 2: Consider a nonlinear second-order system as follows,

$$\begin{aligned}\dot{x}_1 &= \frac{1}{2}x_1^3 + 2x_1x_2^2 + u \\ \dot{x}_2 &= x_2^3 + 3x_1 + \frac{3}{2}x_2 + u\end{aligned} \qquad (16)$$

From (9) we choose $V(x) = p_1x_1^2 + p_2x_2^2 < R$ and from (10) we set

$$\dot{V}(x) = (p_1x_1^2 + p_2x_2^2)(p_1x_1^2 + p_2x_2^2 - R) = h(x)$$

it is seen that $h(x) - \nabla^T V.f(t,x)$ divides $L_g V = x_1 + 2x_2$, if, $p_1 = 1, p_2 = 2$ and $R = 3$. Therefore, from lemma 1 and (13), system (16) is locally exponentially stabilizable for all $x_0 \in E(0, \frac{P}{3})$. where P is a diagonal positive definite matrix as follows

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

and the feedback control is

$$u = -\frac{3}{2}x_1 - 3x_2 \qquad (17)$$

Figures 2(a) and (b) show the system response, with $x_0 = [1.2 \;\; -0.7]$, and the control signal, respectively.

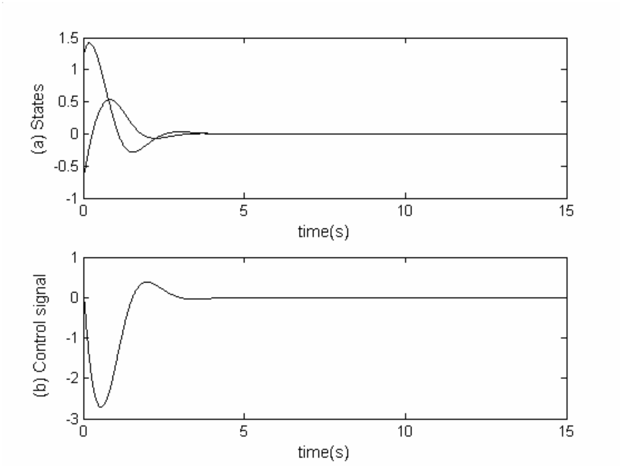

Figure 2. (a) Response of system (16) with $x_0 = [1.2 \;\; -0.7]$ (b) Control signal $u(t)$ for the system defined by (16)

Following is a lemma which proposed for the local uniform exponential stability of (1) with different time derivative of *V*.

Lemma 2: Consider system (1) and assume that origin is an equilibrium state for it. If in a neighborhood *D* of the equilibrium state *x*=0, there exist a differentiable function $V(\cdot) : D \to R$ such that

i. $V(x)$ is positive definite
ii. There exist positive constant α, p such that

$$\alpha \|x(t)\|^p \le V(x) \qquad (18)$$

iii. The derivative of $V(\cdot)$ along any solution of (1) is

$$\dot{V}(x) = -\lambda(\beta - \|x_0\|)V \tag{19}$$

which β, λ are positive constant.

Then (1) is locally uniformly exponentially stable for any $\|x_0\| < \beta$ and its convergence rate is:

$$\frac{\lambda(\beta - \|x_0\|)}{p}.$$

Proof: By defining $\gamma = \beta - \|x_0\|$ so, we have

$$\dot{V} = -\lambda\gamma V$$

and

$$\gamma > 0$$

therefore, $\dot{V}$ is negative definite, clearly system (1) is asymptotically stable, now it needs to show that it is also exponentially stable. It is known

$$\frac{dV}{dt} = -\lambda\gamma V$$

solving the above differential equation leads to

$$V(x) = ke^{-\lambda\gamma t} \tag{20}$$

where, $k = V(0) > 0$.

Equation (18) and (20) leads to

$$\|x(t)\| < \left[\frac{k}{\alpha}\right]^{1/p} e^{-\frac{\lambda\gamma t}{p}} \tag{21}$$

The inequality (21) shows that system (1) is locally uniformly exponentially stable with rate of convergence of $\frac{\lambda\gamma}{p} = \frac{\lambda(\beta - \|x_0\|)}{p}$. □

Now, consider V as equation (18) and time derivative as (19), be a Lyapunov function for system defined by (8). If one set $h(x) = \dot{V}(x) = -\lambda(\beta - \|x_0\|)V$, then (19) leads to

$$h(x) = \nabla^T V . f(t, x) + L_g V u \tag{22}$$

From (22) and lemma 2 it is clear that system defined by (8) is locally exponentially stabilizable for any $x_0 \in B_\beta$ and under control $\varphi(t, x)$ if, $h(x) - \nabla^T V . f(t, x)$ divides $L_g V$, with $\varphi(t, x)$ be the divisor and a continuous function for any $x_0 \in B_\beta$ i.e.

$$\begin{gathered} h(x) - \nabla^T V \cdot f(t, x) = \varphi(t, x) L_g V \\ h(x) - \nabla^T V . f(t, x) \text{ divides } L_g V \\ \varphi(t, x) \text{ is continuous for any } x_0 \in B_\beta \end{gathered} \tag{23}$$

where, the division condition lets using limited control signal. If one choose $\lambda = \frac{k}{\beta - \|x_0\|}$ which k is a positive constant then global exponential stability can be achieved, with rate of convergence of $\frac{1}{p}$. Following are two examples, which illustrate the applicability of the proposed method.

Example 3: Consider a nonlinear first-order system as follows

$$\dot{x} = x^3 - x^2 \sin x - (1 + t)x \cos x + x + u \tag{24}$$

From (18), (19) we choose $V(x) = x^2$ and set $\dot{V}(x) = -(5 - |x_0|)x^2$. It is clear that $h(x) - \nabla^T V . f(t, x)$ divides $L_g V = 2x$. Therefore, from lemma 2 and condition (23), system (24) is locally exponentially stabilizable for any $x_0 \in$ (-5,5). For $x_0 = 2$ the control signal is,

$$u = -\frac{5}{2}x - x^3 + x^2 \sin x + (1 + t)x \cos x \tag{25}$$

The state of the controlled system and the control signal are depicted in Figures 3(a) and (b), respectively.

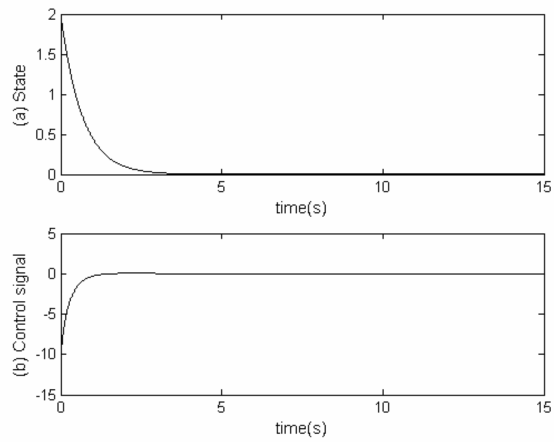

Figure 3. (a) Response of system (24) with $x_0 = 2$
(b) Control signal $u(t)$ for the system defined by (24)

Example 4: Consider a nonlinear non-autonomous system as

$$\begin{aligned} \dot{x}_1 &= -ax_1 + x_n^{m_1} f_1(x_1,\ldots,x_{n-1},x_n,t) \\ &\vdots \\ \dot{x}_{n-1} &= -ax_{n-1} + x_n^{m_{n-1}} f_{n-1}(x_1,\ldots,x_{n-1},x_n,t) \\ \dot{x}_n &= f_n(x_1,\ldots,x_{n-1},x_n,t) + u \end{aligned} \tag{26}$$

where, $m_{i-1} \in R_{\geq 1}$ for $i = 2,\cdots,n$ and $a > 0$. From equation (18) we choose $V(x)$ as follows

$$V(x) = x_1^2 + \cdots + x_n^2 \tag{27}$$

For satisfying condition (23), we should have

$$\|x_0\| = \beta - a \text{ and } \lambda = 2 \tag{28}$$

Therefore, for any arbitrary x_0, by finding β from (28), system (26) from Lemma 2 is exponentially stabilizable. For example consider a third-order system in the form of (26) as follows

$$\begin{aligned} \dot{x}_1 &= -4.5228x_1 - x_3^2 \\ \dot{x}_2 &= -4.5228x_2 + x_3 \cos t \\ \dot{x}_3 &= (1 + \tan^{-1} t)\sin x_1 + x_2^2 - x_3 + u \end{aligned} \tag{29}$$

Here, for example by $x_0 = [1 \ \ {-2} \ \ 5]$, from (28) we find $\beta = 10$. The control law is

$$u = -3.5228x_3 + x_1x_3 - x_2\cos t - (1+\tan^{-1}t)\sin x_1 - x_2^2 \tag{30}$$

The states of the controlled system and the control signal are depicted in Figures 4(a) and (b), respectively.

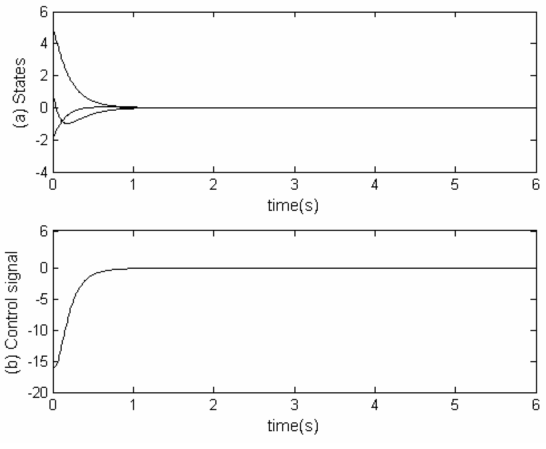

Figure 4. (a) Response of system (29) with $x_0 = [1 \ \ {-2} \ \ 5]$ (b) Control signal $u(t)$ for the system defined by (29)

IV. CONCLUSION

In this paper a novel method for local exponential stabilization for a class of nonlinear non-autonomous systems has been proposed. The convergence rate and convergence region are variable and can be chosen by the designer to satisfy the desired system's performance. The proposed method dose not need to construct Lyapunov functions and a time-invariant one has been introduced. This method can be applied to any first order SISO nonlinear non-autonomous systems if their control input coefficients are constant. For higher order systems checking condition (13) or (23) is necessary. If condition (13) or (23) is satisfied then the proposed method will be more general than Back-stepping and Feedback linearization approach. Further works can be done on more sophisticated nonlinear systems with structural uncertainty.

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