Transfer Matrix Factorization Based Synthesis of Resistively Terminated LC Ladder Networks

Metin Şengül[1] and Zafer Aydoğar[2]

[1,2]Kadir Has University, Istanbul, Turkey
[1]msengul@khas.edu.tr
[2]zaferaydogar@hotmail.com

Abstract

In this paper, a transfer matrix factorization based synthesis algorithm for resistively terminated low-pass LC ladder networks is presented. In the algorithm, component value of the extracted element and the reflection factor of the remaining network are well formulated in terms of reflection factor coefficients of the whole network. An example is presented to exhibit the application of the proposed synthesis algorithm.

1. Introduction

The modern filter synthesis, given a prescribed insertion-loss between a resistive source and a resistive load, is a classical procedure presented in many textbooks on network synthesis [1]-[3]. The method consists of, given the insertion-loss function, determining the squared-magnitude $|\rho(j\omega)|^2$ of the reflection coefficient $\rho(p)$, then getting a stable $\rho(p)$, and, finally, deriving the corresponding driving-point impedance $Z(p)$. From this $Z(p)$, a lossless network terminated by a resistance can be found, satisfying the prescribed insertion-loss.

For some special data, the resulting $Z(p)$ can be developed in continued-fraction expansion, thus yielding a network in ladder form. Some work in the past obtained explicit formulas for the elements in ladder form for some configurations of the poles and zeros of $\rho(p)$ [4]-[8]. For example, Orchard [8] has given explicit formulas for the elements allowing finite frequencies of infinite loss but starting with the driving-point impedance of the unterminated lossless filter.

In this paper, to describe the networks, Belevitch notation of the scattering parameters ($h(p)$, $g(p)$ and $f(p)$ polynomials) are used. All the element values are formulated in terms of the coefficients of these polynomials.

An algorithm to synthesize low-pass LC ladder networks has been proposed. Similar algorithms are going to be developed for high-pass, band-pass and band-stop cases.

2. Low-pass LC Ladder Networks

Consider the ladder networks with inductive series branches and capacitive shunt branches (low-pass) shown in Fig 1. Input impedance of the circuit is $Z(p)$ (input reflection factor is $S(p)$), where p is the usual complex frequency variable ($p=\sigma+j\omega$). The first element is either a series inductor or a shunt capacitor, depending on whether $Z(p)$ or $Y(p)$ has a pole at infinity.

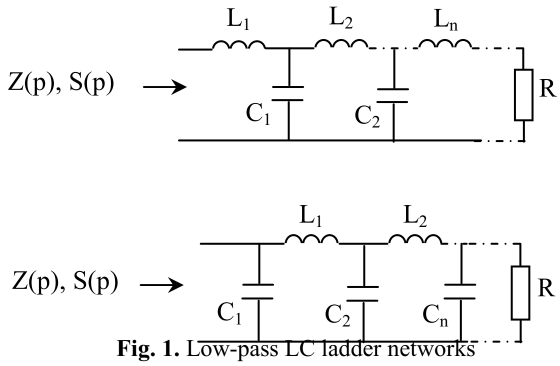

Fig. 1. Low-pass LC ladder networks

Algebraically, the networks just described (first Cauer structure) corresponds to a continuous fraction expansion about the point at infinity,

$$Z(p)=a_1p+\cfrac{1}{a_2p+\cfrac{1}{a_3p+\cfrac{1}{a_4p+\ldots}}}\quad . \tag{1}$$

The a_i are always residues at infinity and can be determined by an iterative long division procedure, in which at each step the remainder is divided into the divisor of the previous step.

The above networks can be described in terms of scattering parameters using the Belevitch notation as follows [9,10];

$$S(p)=\frac{1}{g(p)}\begin{pmatrix} h(p) & \alpha f(-p) \\ f(p) & -\alpha h(-p) \end{pmatrix} \tag{2}$$

where the real polynomials $f(p)$, $g(p)$ and $h(p)$ have the following properties:

- $g(p)$ is a strictly Hurwitz polynomial.
- $f(p)$ is an even or odd polynomial, i.e. $f(-p)=\alpha f(p)$, where $\alpha=\pm 1$.
- $f(p)$, $g(p)$ and $h(p)$ are related by $g(p)g(-p)=h(p)h(-p)+\alpha f^2(p)$

which imposes the following degree relations; $\deg h(p)\le \deg g(p)$, $\deg f(p)\le \deg g(p)$, and the difference $\deg g(p)-\deg f(p)$ specifies the number of transmission zeros at infinity.

For a series inductor, it can be shown that

$h(p)=\frac{L}{2}p$, $g(p)=\frac{L}{2}p+1$ and $f(p)=1$,

and for a shunt capacitor

$h(p)=-\frac{C}{2}p$, $g(p)=\frac{C}{2}p+1$ and $f(p)=1$

where L and C are the element values of the inductor and capacitor, respectively.

Consider low-pass LC ladder network with the first element is either a series inductor or a shunt capacitor. $h(p)$ and $g(p)$ polynomials of each element in the structure can be written as

$$h^{(i)}(p)=(-1)^{i+1}\mu_i G_i p \tag{3}$$

$$g^{(i)}(p)=G_i p+1 \tag{4}$$

for $i=1,2,\ldots,n$, where n is the number of lumped elements, $G_i=\frac{L}{2}$ for a series inductor ($\mu_i=+1$) ,and $G_i=\frac{C}{2}$ for a shunt capacitor ($\mu_i=-1$).

The input reflectance $S(p)$ and the input impedance $Z(p)$ of the ladder network terminated on a unit resistor are expressed in terms of the $h(p)$ and $g(p)$ polynomials as follows;

$$S(p)=\frac{Z(p)-1}{Z(p)+1}=\frac{h(p)}{g(p)}, \tag{5}$$

$$Z(p)=\frac{1+S(p)}{1-S(p)}=\frac{g(p)-h(p)}{g(p)-h(p)}. \tag{6}$$

In the proposed synthesis algorithm, $h(p)$ and $g(p)$ polynomials and $S(p)$ input reflection factor are employed.

Firstly, element values are expressed in terms of $h(p)$ and $g(p)$ polynomial coefficients. Then $h_c^{(k)}(p)$ and $g_c^{(k)}(p)$ polynomials of the extracted element are written. By using these four polynomials, input reflection factor (namely $h_r(p)$ and $g_r(p)$) of the remaining network is obtained, and the next element value is calculated. This process is implemented until all the elements are extracted (Fig 2).

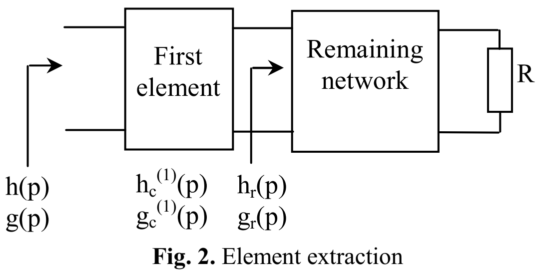

Fig. 2. Element extraction

3. Synthesis via Transfer Matrix Factorization

Cascade decomposition of a lossless two-port network is a classical problem which has been formulated in the literature in many different ways. The conventional approach is to start from a given driving-point function (impedance or reflecrance) and extract elementary sections, depending on the nature of the transmission zeros being extracted. In this approach, the extraction mechanics and the computation of the remaining impedance or reflectance functions can be quite involved and usually require intensive computational operations. An alternative way of accomplishing the canonic decomposition of lossless two-ports in cascade involves factoring the chain matrix or the scattering transfer matrix. It has long been recognized that the transfer matrix constitutes a better tool, mainly because of the simple representation in terms of only three canonical polynomials [11]. The factorization of the transfer matrix of a lossless two-port into a product of two simpler transfer matrices has been treated rigorously by Fettweis [11]. The problem is reduced to the solution of a set of linear equations introducing a mathematically well formulated alternative for the conventional cascade synthesis problem. The methods works directly on the canonic polynomial description of two-ports and involves algebraic decomposition of a given polynomial set, which describes the transfer matrix of a lossless two-port into subsets of polynomials of the same type.

As it is well known, canonic forms of the scattering matrix S and the scattering transfer matrix T of a lossless two-port N , referred to a real terminating resistances are defined as

$$S=\frac{1}{g}\begin{pmatrix} h & \sigma f_* \\ f & -\sigma h_* \end{pmatrix},\quad T=\frac{1}{f}\begin{pmatrix} \sigma g_* & h \\ \sigma h_* & g \end{pmatrix} \tag{7}$$

where (*) represents paraconjugation, g is a strictly Hurwitz polynomial of degree n , and h and f are real polynomials of $\deg rees \le n$ satisfying the paraunitary relation

$$g g_* - h h_* = f f_* . \tag{8}$$

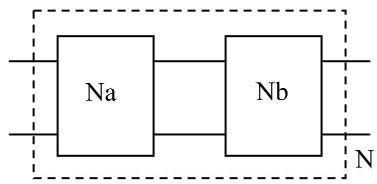

Fig. 3. Cascade decomposition of a lossless two-port

The problem is to decompose the lossless reciprocal two-port N into two cascade connected lossless two-ports N_a and N_b which are also reciprocal (Fig 3). This amounts to factoring the transfer matrix T into a product of two transfer matrices [9],

$$T=T_a\cdot T_b \tag{9}$$

where

$$T_a=\frac{1}{f_a}\begin{pmatrix} \sigma_a g_{a*} & h_a \\ \sigma_a h_{a*} & g_a \end{pmatrix},\quad T_b=\frac{1}{f_b}\begin{pmatrix} \sigma_b g_{b*} & h_b \\ \sigma_b h_{b*} & g_b \end{pmatrix}.$$

The polynomial sets $\{g_a,h_a,f_a\}$ and $\{g_b,h_b,f_b\}$ have the same properties as $\{g,h,f\}$ and in particular must satisfy paraunitary relations similar to (9), i.e.,

$$g_a\, g_{a*} - h_a\, h_{a*} = f_a\, f_{a*}\,, \quad (10a)$$

$$g_b\, g_{b*} - h_b\, h_{b*} = f_b\, f_{b*}\,. \quad (10b)$$

In the proposed method, firstly the component type that will be extracted is determined. Then after calculating the element value, it is extracted and the polynomials of the remaining network have been obtained. This process is repeated until the termination resistance is reached.

3.1. Coefficient Based Component Extraction

Consider the circuits shown in Fig 1, $g(p)$, $h(p)$ and $f(p)$ polynomials of the circuit are as follows

$$g(p) = g_0 + g_1 p + g_2 p^2 + \ldots + g_n p^n \quad (11a)$$

$$h(p) = h_0 + h_1 p + h_2 p^2 + \ldots + h_n p^n \quad (11b)$$

$$f(p) = 1 \quad (11c)$$

Component value of the first element that will be extracted can be calculated as

$$CV = \frac{g_n + \mu\, h_n}{g_{n-1} - \mu\, h_{n-1}} \quad (12)$$

where $\mu = \dfrac{h_n}{g_n}$ and If $\mu = +1$, the first component is an inductor, if $\mu = -1$, the first component is a capacitor.

$g(p)$, $h(p)$ and $f(p)$ polynomials of the remainder can be obtained as

$$h(p) = \sum_{a=1}^{n-1} \left(h_{a+2} \cdot gc_1 + h_{a+1} - g_{a+2} \cdot hc_1 \right) \cdot p^{n-a} \quad (13a)$$

$$h_0 = 0$$

$$g(p) = \sum_{a=1}^{n-1} \left(g_{a+2} \cdot gc_{*1} + g_{a+1} - h_{a+2} \cdot hc_{*1} \right) \cdot p^{n-a} \quad (13b)$$

$$g_0 = 1$$

$$f(p) = 1 \quad (13c)$$

where

$$hc(p) = hc_1\, p + hc_0 = \frac{\mu CV}{2} p \quad (14a)$$

$$hc_*(p) = hc_{*1}\, p + hc_{*0} = -\frac{\mu\, CV}{2} p \quad (14b)$$

$$gc(p) = gc_1\, p + gc_0 = \frac{CV}{2} p + 1 \quad (14c)$$

$$gc_*(p) = gc_{*1}\, p + gc_{*0} = -\frac{2CV}{2} p + 1 \quad (14d)$$

The extraction of the components proceeds in a similar fashion until the final termination resistance is reached.

3.2. Algorithm

Step 1: Enter $h(p)$ and $g(p)$ polynomials as

$h(p) = h_0 + h_1 p + h_2 p^2 + \ldots + h_n p^n$

$g(p) = g_0 + g_1 p + g_2 p^2 + \ldots + g_n p^n$

($h_0 = 0$ and $g_0 = 1$ for a transformerless design)

Set $k = 1$, $m = n$.

Step 2: Set $h^{(k)}(p) = h(p)$, $g^{(k)}(p) = g(p)$ and n (the highest power of the polynomial $h^{(k)}(p)$ or $g^{(k)}(p)$).

Step 3: Define $\mu^{(k)} = \dfrac{h^{(k)}{}_n}{g^{(k)}{}_n}$

If $\mu^{(k)} = +1$, the first component is an inductor, if $\mu^{(k)} = -1$, the first component is a capacitor, and component value can be calculated as

$$CV^{(k)} = \frac{g^{(k)}{}_n + \mu^{(k)} h^{(k)}{}_n}{g^{(k)}{}_{n-1} - \mu^{(k)} h^{(k)}{}_{n-1}}$$

If $k = m$, stop the algorithm.

Step 4: Calculate $h_c{}^{(k)}(p)$ and $g_c{}^{(k)}(p)$ polynomials (their paraconjugate polynomials $h^{(k)}_{c*}(p)$ and $g^{(k)}_{c*}(p)$) of the extracted component, as

$$h_c{}^{(k)}(p) = h^{(k)}_{c1} p + h^{(k)}_{c0} = \frac{\mu^{(k)} CV^{(k)}}{2} p$$

$$h^{(k)}_{c*}(p) = h^{(k)}_{c*1} p + h^{(k)}_{c*0} = -\frac{\mu^{(k)} CV^{(k)}}{2} p$$

$$g_c{}^{(k)}(p) = g^{(k)}_{c1} p + g^{(k)}_{c0} = \frac{CV^{(k)}}{2} p + 1$$

$$g^{(k)}_{c*}(p) = g^{(k)}_{c*1} p + g^{(k)}_{c*0} = -\frac{2CV^{(k)}}{2} p + 1$$

Step 5: Calculate $h(p)$ and $g(p)$ polynomials of the remaining network, as

$$h(p) = \sum_{a=1}^{n-1} \left(h^{(k)}_{a+2} \cdot g^{(k)}_{c1} + h^{(k)}_{a+1} - g^{(k)}_{a+2} \cdot h^{(k)}_{c1} \right) \cdot p^{(n-a)}$$

$$h_0 = 0$$

$$g(p) = \sum_{a=1}^{n-1} \left(g^{(k)}_{a+2} \cdot g^{(k)}_{c*1} + g^{(k)}_{a+1} - h^{(k)}_{a+2} \cdot h^{(k)}_{c*1} \right) \cdot p^{(n-a)}$$

$$g_0 = 1$$

Set $k = k + 1$, and go to Step 2.

4. Illustrative Example

The following polynomials are given for a low-pass LC ladder,

$h(p) = p + 14p^2 - 5p^3 + 60p^4$,

$g(p) = 1 + 7p + 24p^2 + 35p^3 + 60p^4$,

$f(p) = 1$.

Step 1:
$h=[0 \quad 1 \quad 14 \quad -5 \quad 60]$, $g=[1 \quad 7 \quad 24 \quad 35 \quad 60]$, $k=1$ and $m=4$.

Step 2:
$h^{(1)}=[0 \quad 1 \quad 14 \quad -5 \quad 60]$, $g^{(1)}=[1 \quad 7 \quad 24 \quad 35 \quad 60]$, $n=4$.

Step 3: $\mu^{(1)}=\dfrac{h^{(1)}{}_4}{g^{(1)}{}_4}=\dfrac{60}{60}=+1$

Since $\mu^{(1)}=+1$, the first component is an inductor.

$$CV^{(1)}=\frac{g^{(1)}{}_4+\mu^{(1)}h^{(1)}{}_4}{g^{(1)}{}_3-\mu^{(1)}h^{(1)}{}_3}=\frac{60+1\cdot 60}{35-1\cdot(-5)}=3$$

Since $k\neq m$ ($1\neq 4$), go to the next step.

Step 4:

$$h_c{}^{(1)}(p)=h_{c1}^{(1)}p+h_{c0}^{(1)}=\frac{\mu^{(1)}CV^{(1)}}{2}p=\frac{1\cdot 3}{2}p=\frac{3}{2}p$$

$$h_{c*}^{(1)}(p)=h_{c*1}^{(1)}p+h_{c*0}^{(1)}=-\frac{\mu^{(1)}CV^{(1)}}{2}p=-\frac{1\cdot 3}{2}p=-\frac{3}{2}p$$

$$g_c{}^{(1)}(p)=g_{c1}^{(1)}p+g_{c0}^{(1)}=\frac{CV^{(1)}}{2}p+1=\frac{3}{2}p+1$$

$$g_{c*}^{(1)}(p)=g_{c*1}^{(1)}p+g_{c*0}^{(1)}=-\frac{CV^{(1)}}{2}p+1=-\frac{3}{2}p+1$$

Step 5:

$$h(p)=\sum_{a=1}^{3}\left(h_{a+2}^{(1)}\cdot g_{c1}^{(1)}+h_{a+1}^{(1)}-g_{a+2}^{(1)}\cdot h_{c1}^{(1)}\right)\cdot p^{(4-a)}$$

$$=-20p^3+5p^2-\frac{1}{2}p \quad ,h_0=0$$

$$g(p)=\sum_{a=1}^{3}\left(g_{a+2}^{(1)}\cdot g_{c*1}^{(1)}+g_{a+1}^{(1)}-h_{a+2}^{(1)}\cdot h_{c*1}^{(1)}\right)\cdot p^{(4-a)}$$

$$=20p^3+15p^2+\frac{11}{2}p \quad ,g_0=1$$

Set $k=2$ and repeat the procedure as follows.

Step 6:
$h^{(2)}=\left[0 \quad -\frac{1}{2} \quad 5 \quad -20\right]$, $g^{(2)}=\left[1 \quad \frac{11}{2} \quad 15 \quad 20\right]$, $n_p=3$.

Step 7: $\mu^{(2)}=\dfrac{h^{(2)}{}_3}{g^{(2)}{}_3}=\dfrac{-20}{20}=-1$

Since $\mu^{(2)}=-1$, the second component is a capacitor.

$$CV^{(2)}=\frac{g^{(2)}{}_3+\mu^{(2)}h^{(2)}{}_3}{g^{(2)}{}_2-\mu^{(2)}h^{(2)}{}_2}=\frac{20-1\cdot(-20)}{15+1\cdot 5}=2\,.$$

Since $k\neq m$ ($2\neq 4$), continue the process in a similar manner.

After completing the algorithm, the following component values are obtained,
$L_1=3,\quad C_1=2,\quad L_2=5,\quad C_2=4,\quad R=1$.

5. Conclusions

In classical LC ladder synthesis, component values are calculated by using pole-zero removing or continued fraction expansion routines. But in the proposed synthesis algorithm presented here for low-pass ladder LC networks which is based on transfer matrix factorization, all component values are calculated in terms of $h(p)$ and $g(p)$ polynomial coefficients. After extracting an element, reflection factor of the remaining network is expressed, so new $h(p)$ and $g(p)$ polynomials. The same process is repeated until calculating all the elements.

7. References

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