Decomposition of the time delay systems with one delay: the simultaneous similarity of two matrices

Afshin Mesbahi and Mohammad Haeri

Advanced Control System Lab, Department of Electrical Engineering, Sharif University of Technology, Tehran, Iran
mesbahi@ee.sharif.edu, haeri@sina.sharif.edu

Abstract

The time delay equations are frequently used to model real systems since they can describe the after-effect phenomenon. It is usually difficult to analyze the properties of time delay systems. Thus it would be beneficial to make the systems as simple as possible. For this purpose, the decomposition of the systems into some separate or cascaded sub-systems with smaller dimension is introduced as the solution of the mentioned complex task. The decomposition of the time delay systems with one delay is directly related to simultaneous similarity of two matrices. In this regard, necessary and sufficient conditions are derived for the existence of a common similar block diagonal or triangular form for two matrices. Finally, the applications of the obtained results for time delay systems are illustrated via two examples.

1. Introduction

Transmission of information, material, and energy is not instantaneous and therefore time delays appear frequently in real systems. Delay is expressed in different form of models, such as retarded, neutral, and fractional order systems. Each of these models describes behaviour of a special class of real physical systems. Analysis of stability, robust stability, controllability, and observability of these systems are important both from theoretical and practical aspects [1]. Some of these properties are involved with complex computation, where the computation is directly related to their dimension. The stability analysis of these systems can be complex due to the non-deterministic polynomial-time hard (NP-hard) nature of the stability problem [2]. Moreover, some methods that are proposed to analyze these properties are not applicable for the systems with arbitrary dimension [3]. Recently, researchers have been paying more attention to making time delay models as simple as possible. For example, consider differential-different equations with a large number of state variables and multiple low dimensional delay elements. For this kind of systems, time delays can be pulled out from equations, and therefore a simpler form is obtained [4]. The complexity costs of some techniques reduce according to this form [5].

As another problem, unfortunately some techniques and functions can only be employed to analyze the stability of the time delay systems with dimension one [3]. As an example, all stability regions in the delay parameters are determined by using the well-known technique called the cluster treatment of characteristic roots [6]. However, this technique cannot be applied to the case that the characteristic equation of the system has repeated and non-semisimple roots on the imaginary axis. In this case, the system must be decomposed to many separate or cascade sub-systems [7].

The similarity transformation can be employed as a technique to reduce the computational complexity of analysis and to enhance performance of the methods for time delay systems. This topic is a classic topic in linear algebra and control theory for the system without delay. It is much simpler for the system without delay than the system with time delay. The similarity transformations of time delay system relate to simultaneous similarity of the matrices which is classified as wild problem in the matrix theory [8, 9]. A definition of simultaneous similarity for two sets of matrices is given by using the matrix polynomial [10]. The necessary and sufficient conditions for a set of matrices to be simultaneously similar to a set of diagonal or triangular matrices have been presented in [10], where all matrices in the set must be commutative pairwise. Another condition has been offered based on the Lie algebra. The matrices are simultaneously triangularizable if and only if the Lie algebra generated from them is solvable [11]. This statement is known as Lie's theorem, which is frequently employed for analyzing the switched systems [12, 13]. Unfortunately, these theories cannot be employed for block diagonal on triangular form. Recently, other conditions have been proposed to analyze the existence of a similarity transformation matrix that converts the set of matrices to canonical form [14]. However, unfortunately the condition of the theorem relies to the existence of another matrix, and therefore this dependency creates a new problem. Furthermore, some iterative methods have been proposed to check the condition of simultaneously triangularizable form [15].

The purpose of this paper is to develop necessary and sufficient conditions that have been used to check the simultaneous block triangularization or diagonalization of a matrix set. Also, these conditions are used to determine whether a system with single delay can be divided into some separate or cascade sub-systems. The advantages of the proposed method are that the provided conditions only depend on eigenvalues and eigenvectors of the matrices and can be employed to show that whether the system matrices would be simultaneous similar to block diagonalization and triangularization forms. However, these conditions are based on the assumption that each matrix is diagonalizable.

The paper is structured as follows. Section 2 introduces definitions and an assumption on which this paper is based. The main results of the paper are proved in Section 3. The criteria for checking that two matrices can be decomposed to the simultaneously block triangular and diagonal form are presented. In Section 4, numerical examples are given to illustrate the proposed lemmas and theorems. Also applications of decomposition of the time delay system are discussed. Finally Section 5 concludes the paper.

2. Definitions and Assumption

Throughout this paper, the following definitions and assumption are considered.

Definition 1. A diagonal matrix with its main diagonal entries $a_1, a_2, \ldots, a_n$ is described as $diag\left[a_1, a_2, \ldots, a_n\right]$.

Definition 2. The eigenvalues, eigenvectors, and basis for the eigenspace of matrices A and B (where $A, B \in \mathbb{R}^{n\times n}$) are defined as follow.

$$J_A = \left[J_{A_1}, J_{A_2}, \cdots, J_{A_n}\right], \quad AJ_{A_i} = \lambda_i J_{A_i}, \quad i = 1, \ldots, n, \qquad (1)$$

$$J_B = \left[J_{B_1}, J_{B_2}, \cdots, J_{B_n}\right], \quad BJ_{B_i} = \gamma_i J_{B_i}, \quad i = 1, \ldots, n. \qquad (2)$$

Definition 3. Matrices A and B are simultaneously similar to
a) a common block diagonal form consisting of a submatrix of order $r\times r$, if there exists a nonsingular matrix P such that $\tilde{A} = P^{-1}AP = diag\left[\tilde{A}_{11}, \tilde{A}_{22}\right], \tilde{B} = P^{-1}BP = diag\left[\tilde{B}_{11}, \tilde{B}_{22}\right]$.
b) a common block triangular form consisting of a submatrix of order $r\times r$, if there exists a matrix P such that

$$\tilde{A} = P^{-1}AP = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \tilde{B} = P^{-1}BP = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ 0 & \tilde{B}_{22} \end{bmatrix}, \qquad (3)$$

or

$$\tilde{A} = P^{-1}AP = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \tilde{B} = P^{-1}BP = \begin{bmatrix} \tilde{B}_{11} & 0 \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix}, \qquad (4)$$

where $\tilde{A}_{11}, \tilde{B}_{11} \in \mathbb{R}^{r\times r}$, $\tilde{A}_{12}, \tilde{B}_{12} \in \mathbb{R}^{r\times(n-r)}$, $\tilde{A}_{21}, \tilde{B}_{21} \in \mathbb{R}^{(n-r)\times r}$ and $\tilde{A}_{22}, \tilde{B}_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$.

Assumption 1. Matrices A and B are diagonalizable.

3. Main Results

Consider linear time invariant delay systems represented by the following state space model

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \qquad (5)$$

where A and B are given $n\times n$ real constant matrices, $x \in \mathbb{R}^n$ is the state vector, and τ is a positive real number. Many research works have been carried out on the analysis of the systems represented by the above model.

Lemma 1. Consider two $n\times n$ real constant and nonsingular matrices. There exists a nonsingular transformation matrix

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, T_{11} \in \mathbb{R}^{r\times r}, \qquad (6)$$

that transforms the columns (rows) of one matrix to the columns (rows) of another matrix, such that T_{11} and T_{22} are nonsingular matrices for $r = 1, \ldots, n$. □

Proof. Since the transformation matrix is nonsingular, there exists a minor with an arbitrary admissible dimensional such that it is nonsingular. Thus, by using elementary row (column) operations, the claim can be established. ■

From now on it is assumed that the leading principal minors of the transformation matrix are nonzero.

When system (5) is decomposable into two or more separate or cascade sub-systems, since dimension of the sub-systems is less than dimension of the main system then the properties of the main system could be obtained in more efficient ways. Besides, system (5) is decomposable when its matrices are simultaneously similar to a block diagonal or triangular form. Let two matrices are simultaneously similar to the block diagonal or triangular form. Then, there would be several transformation matrices to obtain the desired form. A special kind of these transformation matrices is discussed in the following lemma.

Lemma 2. A necessary and sufficient condition for A and B to be simultaneously similar to
(a) a block upper triangular form consisting of a submatrix of order $r\times r$ is the existence of an eigenspace of B (A) such that

$$\hat{A} = J_B^{-1}AJ_B = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, \hat{B} = J_B^{-1}BJ_B = B_J, \qquad (7)$$

(b) a block lower triangular form consisting of a submatrix of order $r\times r$ is the existence of an eigenspace of B (A) such that

$$\hat{A} = J_B^{-1}AJ_B = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \hat{B} = J_B^{-1}BJ_B = B_J, \qquad (8)$$

(c) a block diagonal form consisting of a submatrix of order $r\times r$ is the existence of an eigenspace of B (A) such that

$$\hat{A} = J_B^{-1}AJ_B = diag\left[\hat{A}_{11}, \hat{A}_{22}\right], \hat{B} = J_B^{-1}BJ_B = B_J, \qquad (9)$$

where $\hat{A}_{11} \in \mathbb{R}^{r\times r}$, $\hat{A}_{12} \in \mathbb{R}^{r\times(n-r)}$, $\hat{A}_{21} \in \mathbb{R}^{(n-r)\times r}$, $\hat{A}_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$ and $B_J = diag\left[\gamma_1, \cdots, \gamma_n\right]$. □

Proof. (a) Suppose A and B are simultaneously decomposable to a block upper triangular form according to Definition 3. Then the proof is followed considering the following linear matrix equation

$$\tilde{B}_{11}R - R\tilde{B}_{22} + \tilde{B}_{12} = 0, \qquad (10)$$

where $R \in \mathbb{R}^{r\times(n-r)}$. Since B is diagonalizable, the columns of $\tilde{B}_{11}$ and the rows of $\tilde{B}_{22}$ span the columns and rows of $\tilde{B}_{12}$ respectively. Therefore (10) has a solution [16]. Matrix $P^{-1}BP$ is diagonalized by the similarity transformation

$$F = \begin{bmatrix} J_{\tilde{B}_{11}} & RJ_{\tilde{B}_{22}} \\ 0 & J_{\tilde{B}_{22}} \end{bmatrix}, \qquad (11)$$

where $J_{\tilde{B}_{11}}$ and $J_{\tilde{B}_{22}}$ are the eigenspaces of $\tilde{B}_{11}$ and $\tilde{B}_{22}$, i.e.,

$$\begin{aligned} F^{-1}P^{-1}BPF &= \begin{bmatrix} J_{\tilde{B}_{11}}^{-1} & -J_{\tilde{B}_{11}}^{-1}R \\ 0 & J_{\tilde{B}_{22}}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ 0 & \tilde{B}_{22} \end{bmatrix} \begin{bmatrix} J_{\tilde{B}_{11}} & RJ_{\tilde{B}_{22}} \\ 0 & J_{\tilde{B}_{22}} \end{bmatrix}. \qquad (12) \\ &= diag\left[\gamma_1, \cdots, \gamma_r, \gamma_{r+1}, \cdots, \gamma_n\right] = B_J. \end{aligned}$$

Let the eigenspace of B be defined by $J_B = PF$. Thus, A is transformed to the block upper triangular form via J_B, i.e.,

$$J_B^{-1}AJ_B=\begin{bmatrix} J_{\tilde{B}_{11}}^{-1} & -J_{\tilde{B}_{11}}^{-1}R \\ 0 & J_{\tilde{B}_{22}}^{-1}\end{bmatrix}\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22}\end{bmatrix}\begin{bmatrix} J_{\tilde{B}_{11}} & RJ_{\tilde{B}_{22}} \\ 0 & J_{\tilde{B}_{22}}\end{bmatrix}=\hat{A}\,,\quad(13)$$

where

$$\begin{aligned}&\hat{A}_{11}=J_{\tilde{B}_{11}}^{-1}\tilde{A}_{11}J_{\tilde{B}_{11}}\,,\hat{A}_{22}=J_{\tilde{B}_{22}}^{-1}\tilde{A}_{22}J_{\tilde{B}_{22}}\,,\\ &\hat{A}_{12}=J_{\tilde{B}_{11}}^{-1}(\tilde{A}_{11}R-R\tilde{A}_{22}+\tilde{A}_{12})J_{\tilde{B}_{22}}.\end{aligned}\qquad(14)$$

The converse of the proof is obtained from Definition 3. Thus the claim is established.

(b) The proof of this part is similar to (a).

(c) It is immediate from (a) where $\hat{A}_{12}=0$. ■

According to Lemma 2, two matrices are simultaneously similar to block diagonal or triangular form based on their eigenspaces. Though, the structure of the eigenspaces is unknown, but the linear transformation between them has a special property. It is easier to check this property than to find a desired eigenspace. This property is introduced in Theorem 1.

Theorem 1. Consider transformation matrix T that linearly transforms a basis for the eigenspace of matrix A (J_A) to a basis for the eigenspace of matrix B (J_B). A and B are simultaneously similar to a common block diagonal form consisting of a submatrix of order $r\times r$ if and only if the i-th row and the j-th column of matrix $T_{21}T_{11}^{-1}$ and the j-th row and the i-th column of matrix $T_{12}T_{22}^{-1}$ are equal to zero while the i-th and the $r+j$-th eigenvalues of A be none identical for $i=1,\ldots,r$ and $j=1,\ldots,n-r$. □

Proof. Let matrices A and B are simultaneously similar to a common block diagonal form consisting of a submatrix of order $r\times r$. Then, there exist matrix T and an eigenspace of matrix B such that satisfy Definition 3, Lemma 1, and $J_B=J_AT$. Based on (9), the following relation is obtained.

$$\begin{aligned}\hat{A}&=T^{-1}J_A^{-1}AJ_AT=T^{-1}diag[\lambda_1,\cdots,\lambda_r,\lambda_{r+1},\cdots,\lambda_n]T\\ &=T^{-1}diag[A_{J_1},A_{J_2}]T=diag[\hat{A}_{11},\hat{A}_{22}],\end{aligned}\qquad(15)$$

where $A_{J_1}=diag[\lambda_1,\cdots,\lambda_r]$ and $A_{J_2}=diag[\lambda_{r+1},\cdots,\lambda_n]$. After multiplying both sides of (15) by T, the following relation is obtained

$$\begin{aligned}&diag\left[A_{J_1},A_{J_2}\right]T-Tdiag\left[\hat{A}_{11},\hat{A}_{22}\right]=\\ &\begin{bmatrix}A_{J_1}T_{11}-T_{11}\hat{A}_{11} & A_{J_1}T_{12}-T_{12}\hat{A}_{22}\\ A_{J_2}T_{21}-T_{21}\hat{A}_{11} & A_{J_2}T_{22}-T_{22}\hat{A}_{22}\end{bmatrix}=0.\end{aligned}\qquad(16)$$

Since T is a linear transformation for two eigenspaces, matrices T_{11} and T_{22} are nonsingular, according to Lemma 1. Let define matrices $\hat{A}_{11}$ and $\hat{A}_{22}$ as follow

$$\hat{A}_{11}=T_{11}^{-1}A_{J_1}T_{11}\,,\qquad(17)$$

$$\hat{A}_{22}=T_{22}^{-1}A_{J_2}T_{22}\,.\qquad(18)$$

From (16), (17), and (18), the following equations are obtained

$$A_{J_2}T_{21}T_{11}^{-1}-T_{21}T_{11}^{-1}A_{J_1}=0\,,\qquad(19)$$

$$A_{J_1}T_{12}T_{22}^{-1}-T_{12}T_{22}^{-1}A_{J_2}=0\,.\qquad(20)$$

These equations are a special form of the Sylvester equation. By using the Kronecker product ($\otimes$) and the vec-operator (Vec), one can get the following relations

$$\left(I_r\otimes A_{J_2}-A_{J_1}^T\otimes I_{n-r}\right)Vec\left(T_{21}T_{11}^{-1}\right)=0\,,\qquad(21)$$

$$\left(I_{n-r}\otimes A_{J_1}-A_{J_2}^T\otimes I_r\right)Vec\left(T_{12}T_{22}^{-1}\right)=0\,,\qquad(22)$$

where I_r is the identity matrix of size r. Due to Assumption 1, these equations are rewritten as follow.

$$\begin{aligned}&diag\left[\lambda_{r+1}-\lambda_1,\cdots,\lambda_n-\lambda_1,\cdots,\right.\\ &\qquad\left.\lambda_{r+1}-\lambda_r,\cdots,\lambda_n-\lambda_r\right]Vec\left(T_{21}T_{11}^{-1}\right)=0,\end{aligned}\qquad(23)$$

$$\begin{aligned}&diag\left[\lambda_1-\lambda_{r+1},\cdots,\lambda_r-\lambda_{r+1},\cdots,\right.\\ &\qquad\left.\lambda_1-\lambda_n,\cdots,\lambda_1-\lambda_n\right]Vec\left(T_{12}T_{22}^{-1}\right)=0.\end{aligned}\qquad(24)$$

Since (23) and (24) must be satisfied, then if the i-th and the $r+j$-th eigenvalues of A be none identical, the i-th row and the j-th column of $T_{21}T_{11}^{-1}$ must be equal to zero. Similarly, the i-th column and the j-th row of $T_{12}T_{22}^{-1}$ must be equal to zero if the i-th and the $r+j$-th eigenvalues of A be none identical.

The reverse part of the proof is explained as follows. A basis for the eigenspace of B is applied to A and B, therefore (9) and (25) are obtained.

$$J_B^{-1}AJ_B=T^{-1}diag\left[A_{J_1},A_{J_2}\right]T=\begin{bmatrix}\hat{A}_{11} & \hat{A}_{12}\\ \hat{A}_{21} & \hat{A}_{22}\end{bmatrix}.\qquad(25)$$

After multiplying both sides of (25) by T, the following relation is obtained.

$$\begin{aligned}&\begin{bmatrix}A_{J_1} & 0\\ 0 & A_{J_2}\end{bmatrix}T-T\begin{bmatrix}\hat{A}_{11} & \hat{A}_{12}\\ \hat{A}_{21} & \hat{A}_{22}\end{bmatrix}\\ &=\begin{bmatrix}A_{J_1}T_{11}-T_{11}\hat{A}_{11}-T_{12}\hat{A}_{21} & A_{J_1}T_{12}-T_{12}\hat{A}_{12}-T_{11}\hat{A}_{12}\\ A_{J_2}T_{21}-T_{21}\hat{A}_{11}-T_{22}\hat{A}_{21} & A_{J_2}T_{22}-T_{22}\hat{A}_{12}-T_{21}\hat{A}_{12}\end{bmatrix}\\ &=\begin{bmatrix}A_{J_1}T_{11}-T_{11}\hat{A}_{11} & A_{J_1}T_{12}-T_{12}\hat{A}_{12}\\ A_{J_2}T_{21}-T_{21}\hat{A}_{11} & A_{J_2}T_{22}-T_{22}\hat{A}_{12}\end{bmatrix}\\ &\qquad-\begin{bmatrix}T_{12} & T_{11}\\ T_{22} & T_{21}\end{bmatrix}\begin{bmatrix}\hat{A}_{21} & 0\\ 0 & \hat{A}_{12}\end{bmatrix}=0.\end{aligned}\qquad(26)$$

$\hat{A}_{11}$ and $\hat{A}_{22}$ are defined in (17) and (18), respectively. Due to Assumption 1 and these definitions, the first matrix in (26) is equal to zero. Moreover, since T is nonsingular, the following relation is achieved.

$$diag\left[\hat{A}_{21},\hat{A}_{12}\right]=0\,.\qquad(27)$$

Thus $\hat{A}_{12}$ and $\hat{A}_{21}$ are $\mathbf{0}$ and the claim is established. ■

If the condition of Theorem 1 is satisfied then the system can be written as many separate sub-systems with small dimensions. Although, the off-diagonal blocks of the transformed matrices are equal to zero for a common block diagonal form, but it is sufficient that one of the off-diagonal blocks be zero for reaching a common triangular form. Hence, the following theorem is straightforwardly obtained from the explained point and Theorem 1.

Theorem 2. Let transformation matrix (6) transforms linearly a basis for the eigenspace of matrix A (J_A) to a basis for the eigenspace of matrix B (J_B). A and B are simultaneously similar to a common triangular form consisting of a submatrix of order $r\times r$ if and only if the i-th row and the j-th column of $T_{21}T_{11}^{-1}$ or the j-th row and the i-th column of $T_{12}T_{22}^{-1}$ be equal to zero while the i-th and the $r+j$-th eigenvalues of A be none identical for $i=1,\ldots,r$ and $j=1,\ldots,n-r$. □

Proof. It is followed by similar arguments of Theorem 1. ■

Since $T_{21}T_{11}^{-1}$ and $T_{12}T_{22}^{-1}$ constitute the conditions of Theorems 1 and 2, therefore $\hat{T}$ is defined to generate these two matrices.

$$\hat{T}=T\begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}^{-1}=\begin{bmatrix} I_r & T_{12}T_{22}^{-1} \\ T_{21}T_{11}^{-1} & I_{n-r} \end{bmatrix}. \qquad (28)$$

Therefore, the conditions of the theorems can be easily checked by a glimpse of zero elements of $\hat{T}$. However, it is not necessary to compute matrix $\hat{T}$ for a special kind of matrices. These cases are introduced in the following corollary.

Corollary 1. If eigenvalues of A or B be equal with each other and it is diagonalizable, then the system can be separated into n sub-systems with dimension one. □

Proof. Since all eigenvalues are equal with each other, there is not any constrains on element of matrices $T_{21}T_{11}^{-1}$ and $T_{12}T_{22}^{-1}$. Therefore, the assumption of Theorem 1 will be satisfied. ■

4. Illustrative Examples

Example 1. Consider system (5) where

$$A=\begin{bmatrix} -\frac{2445}{161}-\frac{2593\sqrt{2}}{161} & -\frac{1717}{161}-\frac{1032\sqrt{2}}{161} & -\frac{3112}{161}-\frac{2064\sqrt{2}}{161} & -\frac{58}{23}-\frac{223\sqrt{2}}{23} \\ -\frac{15}{14}+\frac{23}{7\sqrt{2}} & -\frac{57}{14}+\frac{23}{7\sqrt{2}} & -\frac{36}{7}+\frac{23\sqrt{2}}{7} & 2 \\ \frac{1395}{161}+\frac{1032\sqrt{2}}{161} & \frac{2545}{322}+\frac{503}{161\sqrt{2}} & \frac{2062}{161}+\frac{503\sqrt{2}}{161} & \frac{35}{46}+\frac{223}{23\sqrt{2}} \\ \frac{1801}{322}+\frac{2593}{161\sqrt{2}} & \frac{778}{161}+\frac{516\sqrt{2}}{161} & \frac{1395}{161}+\frac{1032\sqrt{2}}{161} & -\frac{57}{46}+\frac{223}{23\sqrt{2}} \end{bmatrix},$$

$$B=\begin{bmatrix} \frac{2418}{161}+\frac{2626\sqrt{2}}{161} & \frac{19}{7}-\frac{27}{7\sqrt{2}} & \frac{26}{7}-\frac{27}{7\sqrt{2}} & \frac{3433}{161}+\frac{11125}{161\sqrt{2}} \\ -\frac{22}{7}-\frac{27}{14\sqrt{2}} & -\frac{1}{7}-\frac{27}{14\sqrt{2}} & -\frac{22}{7}-\frac{27}{14\sqrt{2}} & -\frac{29}{7}-\frac{27}{14\sqrt{2}} \\ -\frac{864}{161}-\frac{4631}{322\sqrt{2}} & -\frac{5}{7}+\frac{27}{7\sqrt{2}} & \frac{9}{7}+\frac{27}{7\sqrt{2}} & -\frac{1452}{161}-\frac{2626\sqrt{2}}{161} \\ -\frac{1048}{161}-\frac{1313\sqrt{2}}{161} & -\frac{13}{7}+\frac{27}{14\sqrt{2}} & -\frac{20}{7}+\frac{27}{14\sqrt{2}} & -\frac{1153}{161}-\frac{11125}{322\sqrt{2}} \end{bmatrix}.$$

The characteristic equation of the system is $C(s,\tau)=\det(sI_4-A-Be^{-s\tau})$. For $s_0=0.5j$ and $\tau_0=0.5\pi$, since $C(s_0,\tau_0)=0$, $\partial C/\partial\tau|_{v=s_0,\,\tau=\tau_0}=0$ and $\partial C/\partial s|_{v=s_0,\,\tau=\tau_0}=0$, therefore the characteristic equation has a semisimple root at s_0 with multiplicity 2 according to [7]. Thus, the cluster treatment of characteristic roots method which proposed in [6] cannot be employed to analyze the stability of the system, unless the system is decomposed into some separate or cascade sub-systems. The bases for the eigenspace of the matrices are obtained as follow.

$$J_A=\begin{bmatrix} -2 & 0 & -2.1174 & -0.8825 \\ 0.2519 & 1.5597 & 0.1174 & -1.1174 \\ 1.3740 & -0.7798 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$J_B=\begin{bmatrix} -2 & 0 & -2 & -0.2387 \\ -2.9740 & 1.9998 & 0 & 0.2387 \\ 2.9870 & -0.9999 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

It is concluded by substituting of these eigenspaces into (28) that matrix $\hat{T}$ is the identity matrix of size 4. According to the Theorem 1, this system can be decomposed to two sub-systems with dimension 2 by using the linear transformation as $x(t)=J_By(t)$. Thus, the new system, and sub-system 1 and sub-system 2 are determined as follows.

$$\dot{y}(t)=diag\left[\hat{A}_{11},\hat{A}_{22}\right]y(t)+diag\left[\hat{B}_{11},\hat{B}_{22}\right]y(t-\tau), \qquad (29)$$

$$\dot{y}_1(t)=\hat{A}_{11}y_1(t)+\hat{B}_{11}y_1(t-\tau), \qquad (30)$$

$$\dot{y}_2(t)=\hat{A}_{22}y_2(t)+\hat{B}_{22}y_2(t-\tau), \qquad (31)$$

where

$$\hat{A}_{11}=\begin{bmatrix} -3.2389 & 0.9999 \\ -3.2102 & -0.0129 \end{bmatrix}, \hat{A}_{22}=\begin{bmatrix} -10.6167 & -1.8181 \\ -4.1891 & -3 \end{bmatrix},$$

$$\hat{B}_{11}=\begin{bmatrix} 1.5 & 0.5065 \\ 0.5065 & 1.5 \end{bmatrix}, \hat{B}_{22}=\begin{bmatrix} 3 & 4.1891 \\ 4.1891 & 3 \end{bmatrix}.$$

Now, the cluster treatment of the characteristic roots technique is applied for every sub-system to determine the stability of the system. Sub-system 1 has two sets of roots on the imaginary axis as $s_1=1.4089j$, $\tau_{1_k}=-4.3116+4.4593k$ and $s_2=0.5j$, $\tau_{2_k}=0.5(8k\pi-7\pi)$ for $k=1,2,\ldots$. Sub-system 2 has two sets of roots on the imaginary axis as $s_3=s_2,\tau_{3_k}=\tau_{2_k}$ and $s_4=1.5155j,\tau_{4_k}=-0.8882+4.1458k$ for $k=1,2,\ldots$. Therefore, the system is stable for $0<\tau<\tau_{1_1}$. Fig. 1 illustrates these conclusions.

Example 2. Consider the following time delay system [17].

$$\dot{x}(t)=\begin{bmatrix} -3 & -2.5 \\ 1 & 0.5 \end{bmatrix}x(t)+\begin{bmatrix} 1.5 & 2.5 \\ -0.5 & -1.5 \end{bmatrix}x(t-\tau). \qquad (32)$$

The eigenvalues of matrix B are -1 and 1, thus the Assumption 1 is satisfied. Now matrices T and $\hat{T}$ are computed according to (17) as follows

$$T=\begin{bmatrix} 1 & 1 \\ -5 & -1 \end{bmatrix}^{-1}\begin{bmatrix} -2.5 & -1 \\ 1 & 1 \end{bmatrix}\Rightarrow\hat{T}=\begin{bmatrix} 1 & 0 \\ -23/3 & 1 \end{bmatrix}.$$

By considering $r=1$ in Theorem 1, A and B can be reduced to a common triangular form, because $T_{12}T_{22}^{-1}$ is equal to zero. According to Theorem 1, the new system can be obtained from the linear transformation as follows.

$$x(t)=\begin{bmatrix}1 & 1\\ -5 & -1\end{bmatrix}y(t). \tag{33}$$

By using this mapping, the stability analysis of system (32) is equivalent to analysis of the following sub-systems.

$$\dot{y}_1(t)=-2y_1(t)+y_1(t-\tau), \tag{34}$$

$$\dot{y}_2(t)=11.5y_1(t)-0.5y_2(t)-12y_1(t-\tau)-y_2(t-\tau). \tag{35}$$

Then, Lambert w-function can be employed to analyze stability of these sub-systems.

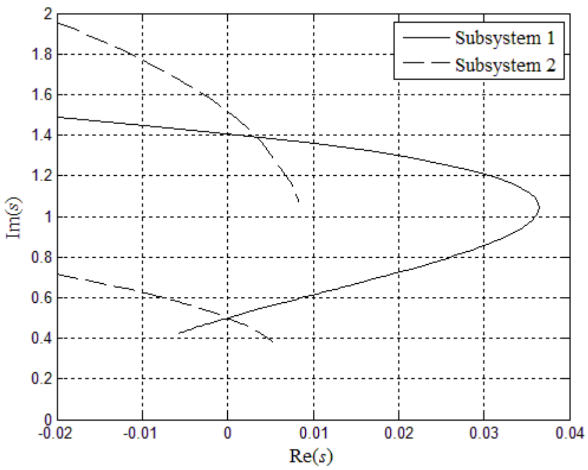

Fig. 1. The root locus of the system near the imaginary axis, for $0<\tau<4.7$.

5. Conclusions

In this paper possibility of the existence of simultaneously block triangular or diagonal forms for two given matrices are investigated. The proposed necessary and sufficient conditions are depended only to the eigenvalues and eigenvectors of the matrices. By using an eigenspace of matrices of a time delay system, the system is divided into some separate and cascade sub-systems with lesser dimension than the original one. The presented Theorems can be extended to set of finite matrices and time delay systems with multiple delays.

6. References

[1] R. Sipahi, T. Vyhlídal, S.I. Niculescu, and P. Pepe, “Time Delay Systems: Methods, Applications and New Trends”, Springer-Verlag, 2012.

[2] O. Toker, and H. Özbay, “Complexity issues in robust stability of linear delay-differential systems”, *Mathematics of Control, Signals, and Systems*, vol. 9, no. 4, pp. 386-400, 1996.

[3] E. Jarlebring, and T. Damm, “The Lambert *w*-function and the spectrum of some multidimensional time-delay systems”. *Automatica*, vol. 43, no. 10, pp. 2124-2128, 2007.

[4] K. Gu, “Stability problem of systems with multiple delay channels”, *Automatica*, vol. 46, no. 4, pp. 743-751, 201.

[5] Y. Zhang, M. Peet, and K. Gu, “Reducing the Complexity of the Sum-of-Squares Test for Stability of Delayed Linear Systems”, *IEEE Transactions on Automatic Control*, vol. 56, no. 1, pp. 229-234, 2011.

[6] N. Olgac, and R. Sipahi, “An exact method for the stability analysis of time delayed LTI systems”, *IEEE Transactions on Automatic Control*, vol. 47, no. 5, pp. 793-797, 2002.

[7] J. Chen, P.L. Fu, S.I. Niculescu, and Z.H, Guan “An eigenvalue perturbation approach to stability analysis, Part II: When will zeros of time-delay systems cross imaginary axis?”, *SIAM Journal of Control and Optimization*, vol. 48, no. 5, pp. 5583-5605, 2010.

[8] L. Hogben, *Handbook of Linear Algebra*, Chapman & Hall/CRC, 2007.

[9] G.R. Belitskii, and V.V. Sergeichuk, “Complexity of matrix problems”, *Linear Algebra and its Applications*, vol. 361, pp. 203-222, 2003.

[10] H. Radjavi, and P. Rosenthal, *Simultaneous Triangularization*, Springer-Verlag, New York, 2000.

[11] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.

[12] D. Liberzon, J.P. Hespanha, and A.S. Morse, “Stability of switched systems: A Lie-algebraic condition", *Systems and Control Letters*, vol. 37, no. 3, pp. 117-122, 1999.

[13] Z. Sun, and S.S. Ge, *Stability Theory of Switched Dynamical Systems*, Springer-Verlag, 2011.

[14] T. Kaczorek, “Similarity Transformation of matrices to one common canonical form and its applications to 2D linear systems”, *International Journal of Applied Mathematics and Computer Science*, vol. 20, no. 3, pp. 507-512, 2010.

[15] J. Zhang, “Simultaneous upper triangularization and the stability of linear discrete multidimensional systems”, *Proceedings of the 7th World Congress on Intelligent Control and Automation*, Chongqing, China, 2008, pp. 513-518.

[16] P. Lancaster, and M. Tismenetsky, *The Theory of Matrices: Second Edition with Applications*, Academic, New York, 1985.

[17] K. Gu, and S.I. Niculescu, “Additional dynamics in transformed time-delay systems”, *IEEE Transactions on Automatic Control*, vol. 45, no. 3, pp. 572-576, 2000.