A ROBUSTNESS ANALYSIS OF AN ENERGY-BASED SWING-UP OF AN INVERTED PENDULUM

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ABSTRACT

The inverted pendulum is a common, interesting control problem that involves many basic elements of control theory. This paper analyzes a particular control system for the swing-up and stabilizing control of an inverted pendulum which is based on energy control. Specifically, robustness of the swing-up control with respect to the system parameter changes is analyzed. Also the design of an acceleration input to the system that restricts the cart motion is investigated. Stabilization phase of the motion is treated by using a pole placement controller.

I. INTRODUCTION

In this paper, we analyze robustness of a particular energy based swing-up method via a set of inverted pendulum motion simulations. Inverted pendulum systems have rich and nonlinear dynamics, and commonly used for testing control strategies ([1], [2]) . They set examples for nonlinear underactuated systems in which pendulum angle and cart position are controlled by a single force input applied to the cart. The problem defined here is to swing-up pendulum from the stable (hang-down) equilibrium point to the unstable (upright) equilibrium point and to stabilize it in its upright position. The swing-up problem has been studied extensively in the literature. We next highlight some of them.

Energy based methods, a popular approach in designing swing-up controllers, are based on injecting energy to the pendulum by applying appropriate control force to the cart. In [3] a bang-bang control is used to raise the energy of the pendulum towards a value equal to its steady state value at the upright position. In [4], a variable structure system version of energy-speed-gradient method is treated in a rigorous manner to show that global attractivity of the upright equilibrium can be achieved by applying a control of arbitrary small magnitude. In [5], the sign condition in the derivative of the energy is exploited. In the paper a servo system having a low pass property is used for the swing-up. This servo system uses a sinusoidal reference input generated from the pendulum trajectory. In another significant energy-based work [1], the swing-up and stabilization of an inverted pendulum system with a restricted cart track length is achieved by using an energy-well built within the cart track. It is constructed in such a way that the cart experiences a repulsive force as it approaches the boundaries in the neighborhood of the limitations. They control the velocity similarly by using a velocity well. In the energy-based works, the stabilization phase is carried out, generally, by using controllers designed for the linearized model of the inverted pendulum

In the next section we present equations of motion for the inverted pendulum system. As a deviation from the work of [1], we do not neglect frictional force between the cart and the surface. It takes place in the equations of motion. Next to writing the state equations for the inverted pendulum, we express the energy of the pendulum in terms its states. In the third section we will introduce the energy based method which we base the robustness analyses on. We use mass of the pendulum as the parametric uncertainty. In the last section, numerical robustness analysis results are given.

II. SYSTEM MODELLING AND STABILIZATION

Inverted pendulum is a two-link robotic system whose motion is restricted to a plane (Figure 1). The control system to be designed is required to control two degrees of freedom (i.e., the pendulum angular position ? and the cart position *x*) by using a single control input, namely the force applied to the cart.

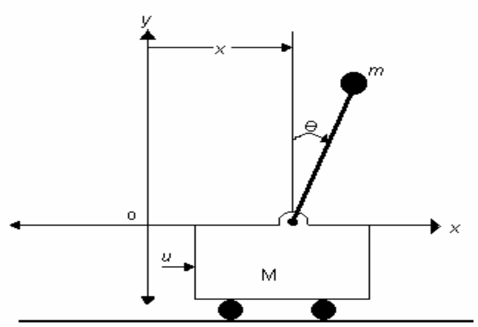

Figure 1: Schematic of an inverted pendulum with cart

Nonlinear model of inverted pendulum system is as follows [6]:

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= \frac{-bx_2 + mlsin(x_3)x_4^2 - mgsin(x_3)cos(x_3) + u}{M + m - mcos(x_3)^2}\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= \frac{(bx_2 - u - mlsin(x_3)x_4^2)cos(x_3) + (M+m)gsin(x_3)}{l(M + m - mcos(x_3)^2)}
\end{aligned} \tag{1}$$

where $x_1 := x$, $x_2 := \dot{x}$, $x_3 := q$, $x_4 := \dot{q}$. Table 1 contains typical parameter values for an inverted pendulum.

Parameters	Symbol	Value	Unit
Mass of the cart	M	3	kg
Mass of the inv. pen.	m	0.5	kg
Length of the inv. pen.	*l*	0.5	m
Friction constant	b	2	kg/s
Gravitional force	g	9.8	m/s^2

Table 1 Typical parameter values for an inverted pendulum.

Using the parameter values given in Table 1, the dynamic model becomes:

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= \frac{-2x_2 + 0.25sin(x_3)x_4^2 - 4.9sin(x_3)cos(x_3) + u}{3.5 - 0.5cos(x_3)^2}\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= \frac{(2x_2 - u - 0.25sin(x_3)x_4^2)cos(x_3) + 34.33sin(x_3)}{1.75 - 0.25cos(x_3)^2}
\end{aligned} \tag{2}$$

The stabilization problem is to design a controller to keep the pendulum in its unstable equilibrium point. In this phase of the problem, we employ a popular control method based on state-space pole-placement design techniques using the linearized model of the inverted pendulum. By this technique, only the local stability, in the vicinity of the upright equilibrium point, is ensured. The pole-placement technique permits the design of a linear controller that achieves arbitrary desired closed loop poles. These poles should be chosen wisely so that the desired closed loop characteristics are achieved. Hence the control law for the stabilization phase is of the form

$$u = -Kx \tag{3}$$

For a physically meaningful design, a small settling time and a high damping ratio are required. To meet these specifications, we place the closed loop poles at $x = \mu_i \ \ (i = 1,2,3,4)$, where $\mu_1 = -1 + j1.732$, $\mu_2 = -1 - j1.732$, $\mu_3 = -4$, and $\mu_4 = -5$. Note that, μ_1 and μ_2 are a pair of dominant closed-loop poles with damping factor $\zeta = 0.5$ and natural frequency $w_n = 2\text{rad/sec}$, resulting in a settling time of approximately 2 to 3 seconds. A feedback controller matrix $K = [-12.24 \quad -13.63 \quad -103.42 \quad -22.31]$ used in (3), meets the desired specifications.

III. ENERGY BASED SWING-UP METHOD

As stated in the preceding sections, the control problem considered in this paper is to move the pendulum to its vertically upright position from its hanging down position. The swing-up routine raises the pendulum to the inverted position, where the linear controller given by (3) can stabilize it. It is crucial that the swing-up routine delivers the pendulum to the inverted position in a controlled, predictable fashion and at small angular velocity [7].

The basic strategy is to move the cart in such a motion that energy is gradually added to the pendulum. Then, a routine is needed to place the cart to the desired position. It is critical that this cart motion is synchronized with the pendulum swings. Due to system disturbances and uncertainties, precalculated movements and pauses does not work. Instead, a control method is needed which reacts to the current system state, and prescribes cart position accordingly.

The method, in this paper, aims at swinging up the pendulum from pendant position to the upright position while maintaining the cart within defined track limits using energy control principles based on the method presented in the paper by Chatterjee et. al. [1]. To achieve this, an “energy well” is built within the cart track not only to prevent the cart from going outside the limited length, but also to limit the velocity of the cart from exceeding a certain value. In addition to that, the swing up routine is done with the goal of attaining a defined energy value, and when this sufficient energy is acquired by the pendulum it goes into a cruise mode as long as the acquired energy is maintained. In this approach, a control strategy is employed by introduction of “potential wells” for the cart position, and for effective control within the cart track length restrictions. Within that well, energy is injected into the system in such a way as to drive the potential and rotational kinetic energy towards a value that is equal to the potential energy of the pendulum in the upright position. In this process the oscillation of the cart is kept under control by introducing penalties on the cart velocity. The energy required to keep the pendulum at the upright position needs to be maintained after it is acquired since we do not have direct control on the configuration at the instant when this energy is reached. The system is controlled now by the energy maintenance mode that gives rise to the name the cruise mode.

The sum of the rotational kinetic energy of the pendulum and its potential energy, denoted by V, is

$$V = \frac{1}{2} ml^2 x_4^2 + mgl\cos(x_3) \quad (4)$$

And, the desired maximum energy for pendulum up position to be

$$V_{up} = mgl \quad (5)$$

Let us define acceleration input component which suits the definition of the control law

$$\mathrm{U}_{\text{cart well}} = \mathrm{K}_{\text{cw}} \operatorname{sgn}(x_1) \log\left(1 - \frac{|x_1|}{L}\right) \quad (6)$$

where K_{cw} parameter controls the cart motion within restriction $[-L, L]$, where L is the track length of the pendulum system. The $\mathrm{U}_{\text{cart well}}$ component will force the cart to change its course when its approaches the limit $[-L, L]$. Then, we define another acceleration input component which also obeys the definition of the control law

$$\mathrm{U}_{\text{velocity well}} = \mathrm{K}_{\text{vw}} \operatorname{sgn}(x_2) \log\left(1 - \frac{|x_2|}{\mathrm{V}_{\max}}\right) \quad (7)$$

where K_{vw} parameter controls the velocity of the cart, however, this parameter is more experimental, meaning that, when implementing the pendulum design in the laboratory, the condition should be provided so that the velocity of the cart never exceeds given limit $\mathrm{V}_{\max}$ during swing-up of the pendulum. The $\mathrm{U}_{\text{velocity well}}$ component will force the cart to change its course when the velocity of the cart, $|x_2|$, approaches the limit $\mathrm{V}_{\max}$.

Lastly, we define another acceleration input component which defines the energy condition for the system

$$\mathrm{U}_{\text{energy maint}} = \mathrm{K}_{\text{em}} \left(\exp|V - NV_{up}| - 1\right) \operatorname{sgn}(V - V_{up}) \operatorname{sgn}(x_4 \cos x_3) \quad (8)$$

where K_{em}, is the energy controlling parameter of the input. This parameter is crucial in order to achieve swing-up since alteration of this value changes the rate of energy injection into the system. And, the other parameter, *N*, ensures the stability in Lyapunov sense in the cruise mode, and *N* should be greater than one (the proof takes place in in reference [1, section 6.2].)

When we combine all the acceleration components we come up with the final expression for the acceleration input to the cart, this satisfies all the necessary conditions for the system to reach the cruise mode

$$\ddot{x} = \mathrm{U}_{\text{cart well}} + \mathrm{U}_{\text{velocity well}} + \mathrm{U}_{\text{energy-maint}} \quad (9)$$

IV. OVERALL ALGORITHM AND SWING-UP ROBUSTNESS TESTS

There should be an intermediate algorithm to switch between the swing-up controller and stabilization controller depending on the state variables x_3 and x_4. This transition should be smooth; one which does not upset the system due to parameter uncertainties and unmodelled dynamics. Principally, when x_3 and x_4 are close to zero, only the stabilization controller is used. Otherwise, when x_3 and x_4 deviates significantly from zero, only the swing-up controller algorithm is used. It should observe that, even though the stabilization algorithm eventually takes the x_3 value to zero, it still continues to take effect until all the state variables are stabilized to zero.

We verify validity of the outlined approach by simulating the motion of the inverted pendulum for various initial conditions. The resulting graphics are depicted in Figures 2 and 3. In the simulations we use the nominal design parameters $\mathrm{K}_{\text{cw}} = 2.5$, $\mathrm{K}_{\text{vw}} = 15$, $\mathrm{K}_{\text{em}} = 0.01$, L=0.5, and $\mathrm{V}_{\max} = 7\,\mathrm{m/s}$.

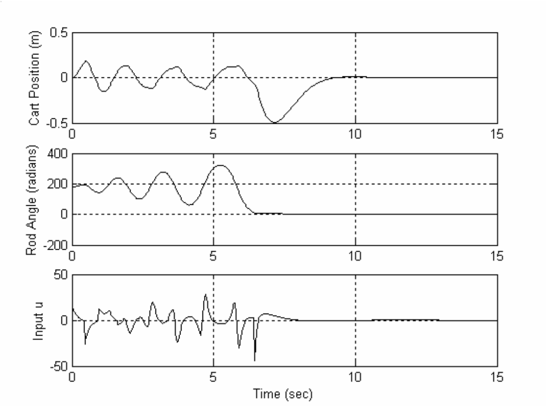

Figure 2**:** Motion simulation from pendant position to the upright position

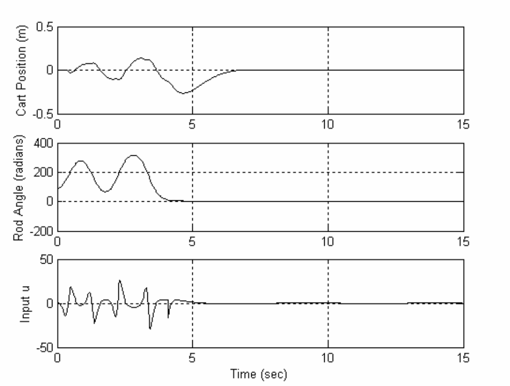

Figure 3: Motion simulation for the initial angular position 90^0

The simulations above show successful behavior of the algorithm for different initial conditions. Indeed, the swing-up goal is achieved successfully for all values of x_3 between 0 and 180 degrees. Also, the plots (Figures 2-3) clearly demonstrates the effectiveness of the control law for the designed potential wells on the input acceleration which enables swing-up and eventually maintains the pendulum in the cruise mode.

As a recapitulation, note that our system model considers the translational coefficient of frictional force between the cart and the surface. In [1], it is neglected; consequently, the number of swings is affected. This paper covers the analysis of the robustness of the system for changes in swing-up controller parameters and the pendulum mass, *m* separately. Since, there is no valid analytic method for analysis of robustness of system for nonlinear swing-up control, all robustness analysis are carried out through simulations.

Firstly, system parameters are fixed to nominal values given by table 1, and various selection of swing-up controller design parameters are tested while keeping the initial conditions the same. When swing-up controller parameters K_{cw}, K_{vw} and K_{em} vary between the intervals $[1.5, 7]$, $[10, 30]$ and $[0.01, 0.035]$ respectively, swing-up is achieved. One sample simulation graphic is shown in Figure 4.

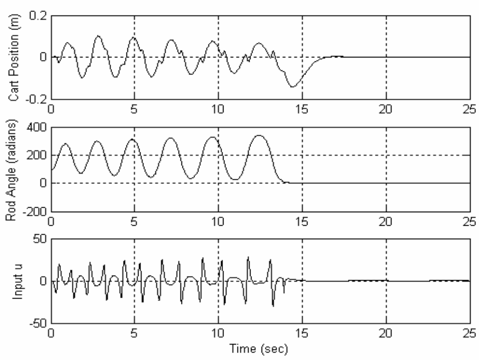

Figure 4: $K_{cw} = 7$, $K_{vw} = 15$, $K_{em} = 0.01$

Secondly, nominal swing-up controller design parameters $K_{cw} = 2.5$, $K_{vw} = 15$, $K_{em} = 0.01$, L=0.5, and $V_{max} = 7\text{m/s}$ are fixed, and the system parameter *m* is changed. Note that, the mass of the pendulum in the system was originally 0.5 kg. For the same controller parameters the swing-up is achieved up to the mass of 0.65 kg. One sample simulation graphic is shown in Figure 5.

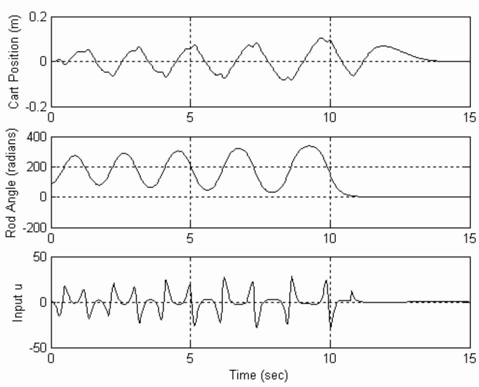

Figure 5: Nominal swing-up controller parameters and mass $m = 0.65$

V. CONCLUSION

The swing-up problem of an inverted pendulum has been treated by using energy control. Robustness with respect to the swing-up controller parameters and the pendulum mass has been investigated. It has been shown that proper selection of K_i parameters results in better robustness margins. The stabilization by using state-feedback pole placement has been achieved and incorporated in the overall control algorithm. The motion simulations have been presented for various K_i parameters. Also we use the “well” concept to maintain the movement of the cart within the defined limits.

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