ESTIMATION OF CLOSE SIGNALS IN GAUSSIAN AUTO-REGRESSIVE NOISE: WORST AND BEST CASE CRAMÉR-RAO BOUNDS

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ABSTRACT

Non-matrix expressions are presented for the Cramér-Rao (CR) lower bound on frequency estimation variance for two closely spaced sinusoidal signals observed in Gaussian auto-regressive noise for both real and complex data cases. The expressions give the dependence of the CR bound on the phase difference between the two signals explicitly and allow one to determine the largest and the smallest values of the bound and the corresponding critical values of the phase difference.

I. INTRODUCTION

The issue of frequency estimation is considered for two-signal time-series data models consisting of either a single real sinusoid in real Gaussian auto-regressive (AR) noise (the real model) or two complex sinusoids in complex Gaussian AR noise (the complex model). These models extensively are used for testing the performance of the so-called high-resolution frequency estimators where the separation of the two signal frequencies present in each model is assumed to be less than the resolution limit of the periodogram (the Fourier limit).

It is well known that the Cramér-Rao (CR) bound specifies a lower bound on the variance of unbiased estimators of signal parameters. The CR bound, being independent of the estimation algorithm used, is frequently studied to obtain the ultimate performance limits for data models.

Expressions for the CR bound for estimating the frequency parameters of the foregoing models are well documented (e.g., see [1-3]). However, these expressions do not show the dependence of the bound on the signal phases explicitly. It is known that the bound strongly depends on the phase difference between the two signal components of the models when the frequency separation between the two signals is less than the Fourier limit. For this small-frequency-separation regime, it thus becomes important to determine the largest and the smallest values of the bound and the corresponding critical values of the phase difference.

A recent paper [4] treated the problem for the case of white Gaussian noise. An analytical solution to the problem requires a non-matrix expression for the bound which gives the dependence of the bound on the phase difference explicitly. Moreover, this dependence should be simple enough to allow an analytical determination of the critical phase differences. This paper extends the non-matrix bound formulae in [4] to the colored Gaussian AR noise case. It is shown that the simple dependence of the CR frequency bound on the phase difference also holds for the colored noise case.

II. THE REAL MODEL CR FREQUENCY BOUND

We first consider the real model which consists of a single real sinusoid in real Gaussian AR noise:

$$y(t) = \alpha_0 \cos(\omega_0 t + \varphi_0) + e(t)\ ,\ \ t = t_0, t_1, \dots, t_{N-1} \qquad (1)$$

where α_0 is the amplitude, ω_0 is the frequency, φ_0 is the phase of the sinusoid, $t_0 < t_1 < \cdots < t_{N-1}$ are the sampling instants, N is the total number of data samples and $e(t)$ is a real Gaussian $\mathrm{AR}(p)$ noise obtained by filtering real white Gaussian noise $w(t)$ with mean zero and variance σ_w^2 through a pth-order all-pole filter:

$$e(t_n) = -\sum_{i=1}^{p} a_i e(t_{n-i}) + w(t_n)\ .$$

We assume that the sampling instants (the t_n's) and the order p of the AR noise are known but all the other parameters of the model are unknown and concentrate on the estimation of the frequency parameter ω_0.

If $\hat{\omega}_0$ is an unbiased estimator of the ω_0 calculated from the N data samples $y(t_0), y(t_1), \dots, y(t_{N-1})$, then the variance of the estimator satisfies the CR theorem:

$$\mathrm{var}(\hat{\omega}_0) \geq B_0$$

where B_0 is the diagonal term corresponding to the parameter ω_0 of the inverse of the Fisher information matrix (FIM) for the problem. The following proposition gives a non-matrix expression for the bound B_0 that reveals the dependence of the bound on the phase φ_0 of the sinusoid.

Proposition 1: Let the data be given by (1). Then the CR frequency bound B_0 can be expressed as

$$B_0 = \frac{2\sigma_w^2}{\alpha_0^2 N^3} \frac{1}{K_0 + M_0 \cos(2\varphi_0) + N_0 \sin(2\varphi_0)} \quad (2)$$

where

$$K_0 = c_1 - \frac{A_1(b_3^2 + b_4^2) - 2A_2(b_1 b_3 + b_2 b_4) + A_3(b_1^2 + b_2^2)}{2(A_1 A_3 - A_2^2)}$$

$$M_0 = c_2 - \frac{A_1(b_3^2 - b_4^2) - 2A_2(b_1 b_3 - b_2 b_4) + A_3(b_1^2 - b_2^2)}{2(A_1 A_3 - A_2^2)}$$

$$N_0 = c_3 - \frac{A_1 b_3 b_4 - A_2(b_1 b_4 + b_2 b_3) + A_3 b_1 b_2}{A_1 A_3 - A_2^2}$$

with $A_{i_A} = a^T \Gamma_{A_{i_A}} a$, $i_A = 1,2,3$, $b_{i_b} = a^T \Gamma_{b_{i_b}} a$, $i_b = 1,2,3,4$, $c_{i_c} = a^T \Gamma_{c_{i_c}} a$, $i_c = 1,2,3$ where $a = (1, a_1, a_2, \ldots, a_p)^T$ and the (k,l)th elements of the Γ matrices are

$$\Gamma_{A_1}(k,l) = N^{-1} \sum_n 2 \cdot \cos(\omega_0 t_{n+k}) \cdot \cos(\omega_0 t_{n+l})$$

$$\Gamma_{A_2}(k,l) = N^{-1} \sum_n -2 \cdot \sin(\omega_0 t_{n+k}) \cdot \cos(\omega_0 t_{n+l})$$

$$\Gamma_{A_3}(k,l) = N^{-1} \sum_n 2 \cdot \sin(\omega_0 t_{n+k}) \cdot \sin(\omega_0 t_{n+l})$$

$$\Gamma_{b_1}(k,l) = N^{-2} \sum_n -2 \cdot t_{n+k} \cdot \sin(\omega_0 t_{n+k}) \cdot \cos(\omega_0 t_{n+l})$$

$$\Gamma_{b_2}(k,l) = N^{-2} \sum_n -2 \cdot t_{n+k} \cdot \cos(\omega_0 t_{n+k}) \cdot \cos(\omega_0 t_{n+l})$$

$$\Gamma_{b_3}(k,l) = N^{-2} \sum_n 2 \cdot t_{n+k} \cdot \sin(\omega_0 t_{n+k}) \cdot \sin(\omega_0 t_{n+l})$$

$$\Gamma_{b_4}(k,l) = N^{-2} \sum_n 2 \cdot t_{n+k} \cdot \cos(\omega_0 t_{n+k}) \cdot \sin(\omega_0 t_{n+l})$$

$$\Gamma_{c_1}(k,l) = N^{-3} \sum_n t_{n+k} \cdot t_{n+l} \cdot \cos(\omega_0 (t_{n+k} - t_{n+l}))$$

$$\Gamma_{c_2}(k,l) = N^{-3} \sum_n -t_{n+k} \cdot t_{n+l} \cdot \cos(\omega_0 (t_{n+k} + t_{n+l}))$$

$$\Gamma_{c_3}(k,l) = N^{-3} \sum_n t_{n+k} \cdot t_{n+l} \cdot \sin(\omega_0 (t_{n+k} + t_{n+l}))$$

Here, $0 \leq k, l \leq p$ and the summation over n extends from 0 to $N-1-k-l$. Also, it is assumed that the number of data samples N satisfies the relation $N > 2p$.

Proof: The proof begins with an expression for the applicable FIM (e.g., see [1], [3]) and proceeds along lines similar to those of the proof for the white noise case given in [4, Appendix A]. Specifically, the decomposition technique introduced in [4] is employed to reveal the phase dependence of the bound. Also, the elements of the FIM are computed via the recently proposed fast algorithm [5, Lemma 1].

Equation (2) is the non-matrix expression for the CR frequency bound for the real model in the case of colored Gaussian AR noise. Note that the quantities K_0, M_0 and N_0 in (2) do not depend on the signal phase φ_0. In the case of white Gaussian noise, the AR parameters a_i, $i = 1,2,\ldots,p$, are all zero and only the (0,0)th terms of the Γ matrices enter into the calculations. The expression in (2) then reduces to that in [4, Equation (5)].

Expression (2) is quite useful in that it gives the dependence of the CR frequency bound B_0 on the phase φ_0 of the sinusoid in a simple way. Note that the bound is periodic in φ_0 with a period of π and, hence, it is sufficient to consider the bound in the interval $\Phi_0 = \left\{\varphi_0 : \varphi_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}$. The largest and the smallest values of the bound B_0 and the corresponding critical values of the phase φ_0 can easily be determined from (2) as described by the following corollary (also given in [4]).

Corollary 1: The CR frequency bound B_0 considered as a function of the phase φ_0 has one maximum point and one minimum point in the interval Φ_0 given by, respectively

$$(\varphi_0)_{\max} = \begin{cases} (\varphi_0)_c - \mathrm{sgn}\left(\frac{N_0}{M_0}\right) \cdot \frac{\pi}{2} & \text{if } M_0 \geq 0 \\ (\varphi_0)_c & \text{if } M_0 < 0 \end{cases}$$

$$(\varphi_0)_{\min} = \begin{cases} (\varphi_0)_c & \text{if } M_0 \geq 0 \\ (\varphi_0)_c - \mathrm{sgn}\left(\frac{N_0}{M_0}\right) \cdot \frac{\pi}{2} & \text{if } M_0 < 0 \end{cases}$$

where

$$(\varphi_0)_c = \frac{1}{2} \arctan\left(\frac{N_0}{M_0}\right)$$

$$\arctan(\cdot) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$

Furthermore, maximum and minimum values of B_0 are given by, respectively

$$(B_0)_{\max} = \frac{2\sigma_w^2}{\alpha_0^2 N^3} \frac{1}{K_0 - \sqrt{M_0^2 + N_0^2}}$$
$$(B_0)_{\min} = \frac{2\sigma_w^2}{\alpha_0^2 N^3} \frac{1}{K_0 + \sqrt{M_0^2 + N_0^2}}.$$

Example 1: Consider the real model in (1) with $N = 10$ data samples and uniform sampling times $t_n = n$, $n = 0,1,...,N-1$. Let the noise be an AR(2) process with a double pole at 0.8 (corresponding to a narrow band noise spectrum having its peak at zero). Figure 1 shows the largest and the smallest values of the CR frequency bound B_0. The vertical co-ordinate in the Figure depicts the value of the product $B_0 \cdot \mathrm{SNR}_0 \cdot N^3$ where SNR_0 denotes the local signal-to-noise ratio (i.e., the ratio of the signal power to the noise power at the signal frequency). The horizontal co-ordinate in the Figure depicts the value of $\delta\omega/\Omega$ where $\delta\omega$ denotes the separation of the two signal frequencies present in the real model, $\delta\omega = 2 \cdot \omega_0$, and Ω denotes the Fourier limit, $\Omega = 2\pi/N$. We see that the difference between the two limits of the bound is large in the interval $\delta\omega/\Omega < 1$ indicating the strong dependence of the bound on the phase of the sinusoid in this region. For $\delta\omega/\Omega > 1$, the difference becomes small and the dependence of the bound on the phase may be neglected. Figure 2 shows the worst-case and the best-case values of the phase versus the normalized frequency difference $\delta\omega/\Omega$ for the interval $\delta\omega/\Omega < 1$.

III. THE COMPLEX MODEL CR FREQUENCY BOUND

The complex counterpart of the previous model consists of two complex sinusoids in a complex Gaussian AR(p) noise:

$$y(t) = \sum_{i=1}^{2} \alpha_i \exp\{j(\omega_i t + \varphi_i)\} + e(t)\,,\; t = t_0, t_1, ..., t_{N-1} \quad (3)$$

where the noise $e(t)$ is obtained by filtering complex white Gaussian noise with mean zero and variance σ_w^2 through a pth-order all-pole filter with (possibly) complex coefficients (the a_i's).

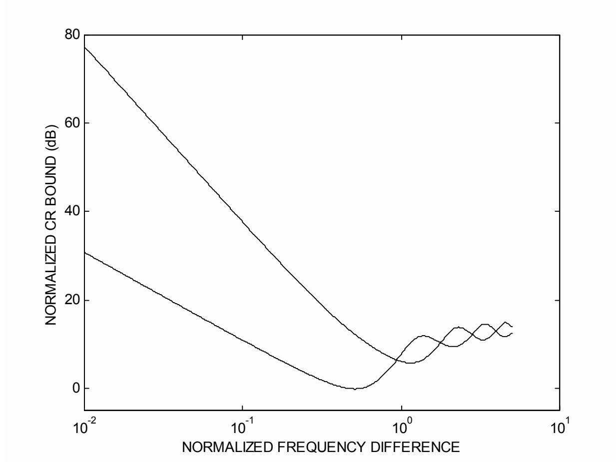

Figure 1. Largest and smallest normalized CR bounds for estimating the frequency of one real sinusoid in a real Gaussian AR(2) noise as a function of the frequency separation. The noise has a double pole at 0.8. Ten uniformly spaced samples are taken.

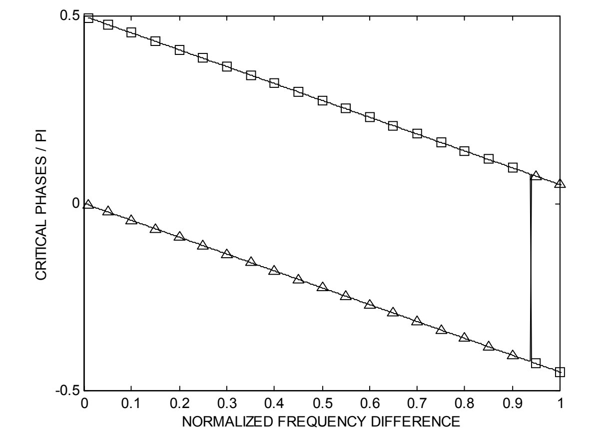

Figure 2. Worst-case (squares) and best-case (triangles) values of the phase of one real sinusoid in a real Gaussian AR(2) noise as a function of the frequency separation. The noise has a double pole at 0.8. Ten uniformly spaced samples are taken with the first sampling instant at zero.

Let B_i denote the CR bound on the variance of unbiased estimators of the frequency parameter ω_i, $i = 1, 2$. The following proposition gives the bound B_i as a simple function of the phase parameters φ_1 and φ_2.

Proposition 2: Let the data be given by (3). Then the CR frequency bounds B_i, $i = 1, 2$, are given by

$$B_i = \frac{\sigma_w^2}{\alpha_i^2 N^3} \frac{1}{K_i + M_i \cos(2(\varphi_1 - \varphi_2)) + N_i \sin(2(\varphi_1 - \varphi_2))} \quad (4)$$

where

$$K_i = \frac{2z^{(1)}z^{(2)} - \left|z^{(3)}\right|^2}{z^{(i)}}$$

$$M_i = \frac{(\operatorname{Im} z^{(3)})^2 - (\operatorname{Re} z^{(3)})^2}{z^{(i)}}$$

$$N_i = -\frac{2\operatorname{Re} z^{(3)} \operatorname{Im} z^{(3)}}{z^{(i)}}$$

$$z^{(1)} = z_3 - \frac{X_1|y_4|^2 - 2\operatorname{Re}(X_2 y_2^* y_4) + X_3|y_2|^2}{X_1 X_3 - |X_2|^2}$$

$$z^{(2)} = z_1 - \frac{X_1|y_3|^2 - 2\operatorname{Re}(X_2 y_1^* y_3) + X_3|y_1|^2}{X_1 X_3 - |X_2|^2}$$

$$z^{(3)} = z_2 - \frac{X_1 y_3^* y_4 - X_2 y_1^* y_4 - X_2^* y_2 y_3^* + X_3 y_1^* y_2}{X_1 X_3 - |X_2|^2}$$

with $X_{i_X} = a^{*^T} \Gamma_{X_{i_X}} a$, $i_X = 1, 2, 3$, $y_{i_y} = a^{*^T} \Gamma_{y_{i_y}} a$, $i_y = 1, 2, 3, 4$, $z_{i_z} = a^{*^T} \Gamma_{z_{i_z}} a$, $i_z = 1, 2, 3$ (the superscript "$*$" denotes the conjugate) where $a = (1, a_1, a_2, ..., a_p)^T$ and the (k,l) th elements of the Γ matrices are

$$\Gamma_{X_1}(k,l) = N^{-1} \sum_n \exp\{j(\omega_1(t_{n+k} - t_{n+l}))\}$$

$$\Gamma_{X_2}(k,l) = N^{-1} \sum_n \exp\{j(\omega_2 t_{n+k} - \omega_1 t_{n+l})\}$$

$$\Gamma_{X_3}(k,l) = N^{-1} \sum_n \exp\{j(\omega_2(t_{n+k} - t_{n+l}))\}$$

$$\Gamma_{y_1}(k,l) = N^{-2} \sum_n j \cdot t_{n+k} \cdot \exp\{j(\omega_1(t_{n+k} - t_{n+l}))\}$$

$$\Gamma_{y_2}(k,l) = N^{-2} \sum_n j \cdot t_{n+k} \exp\{j(\omega_2 t_{n+k} - \omega_1 t_{n+l})\}$$

$$\Gamma_{y_3}(k,l) = N^{-2} \sum_n j \cdot t_{n+k} \cdot \exp\{j(\omega_1 t_{n+k} - \omega_2 t_{n+l})\}$$

$$\Gamma_{y_4}(k,l) = N^{-2} \sum_n j \cdot t_{n+k} \cdot \exp\{j(\omega_2(t_{n+k} - t_{n+l}))\}$$

$$\Gamma_{z_1}(k,l) = N^{-3} \sum_n t_{n+k} \cdot t_{n+l} \cdot \exp\{j(\omega_1(t_{n+k} - t_{n+l}))\}$$

$$\Gamma_{z_2}(k,l) = N^{-3} \sum_n t_{n+k} \cdot t_{n+l} \cdot \exp\{j(\omega_2 t_{n+k} - \omega_1 t_{n+l})\}$$

$$\Gamma_{z_3}(k,l) = N^{-3} \sum_n t_{n+k} \cdot t_{n+l} \cdot \exp\{j(\omega_2(t_{n+k} - t_{n+l}))\}$$

Here, $0 \le k, l \le p$, the summation index n runs from 0 to $N - 1 - k - l$ and the relation $N > 2p$ is assumed to hold.

Proof: The proof is similar to the proof for the white noise case given in [4, Appendix C] with the FIM replaced by the FIM for the colored AR noise case (given in, e.g., [2], [3]). Additionally, here we have used an extension of [5, Lemma 1] to the complex noise case in computing the elements of the FIM.

Equation (4) is the non-matrix expression for the CR frequency bound for the complex model in the case of colored Gaussian AR noise. Note that the bound B_i depends on the two signal phases through their difference. The expression in (4) can be used to determine the largest and the smallest values of the bound and the corresponding critical values of the phase difference.

IV. CONCLUSIONS

We have presented non-matrix expressions for the CR frequency bound for two-signal time-series data models consisting of one real or two complex sinusoids in Gaussian AR noise. The expressions explicitly show the dependence of the bound on the phase difference between the two signal components of each model. These expressions can be used to determine the largest and the smallest values of the bound and the corresponding critical values of the phase difference. This may be of interest when the separation between the two signal frequencies is small.

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