Algebraic Connectivity Approach to Formation Reconfiguration in Multi-agent Systems

Hossein Barghi Jond[1], Vasif V. Nabiyev[1], and Nurhan Gürsel Özmen[2]

[1]Department of Computer Engineering, Karadeniz Technical University, Trabzon, Turkey
barghi@ktu.edu.tr, vasif@ktu.edu.tr
[2]Department of Mechanical Engineering, Karadeniz Technical University, Trabzon, Turkey
gnurhan@ktu.edu.tr

Abstract

In this paper, formation reconfiguration for a multi-agent system is considered wherein a group of agents should move to a desired formation. Agents are restricted with neighborhood connections and also the maximum size of connection links among agents is limited. To obtain a well-structured topology for desired reconfiguration process considering the constraints, graph algebraic connectivity is used. A graph optimization problem is formed to obtain the communication topology with the largest algebraic connectivity in desired formation. An algorithm is presented to solve the optimization problem. After the well topology is determined, reconfiguration is accomplished with force the agents to move desired positions. To this end, the agents control problem can be formed as differential game, and the open-loop Nash equilibrium can be applied as formation control strategy. An example with four agents verified that the formation communication topology with the largest algebraic connectivity is well in order of accomplish formation with less distance error.

1. Introduction

This paper deals with the follow problem: a networked multi-agent autonomous system should be change its initial formation to a desired. Each agent can see only its neighbors in both initial and desired formation. When the formation reconfiguration is accomplished, in the desired formation each agent's neighborhood maybe change. Each agent and its interaction with the neighbors is represented by a node and edges at an undirected graph. Since that interaction network has restricted resources, it can support a limited number of inter-agent connections. In graph representation, this means a graph with bounded edge number. The networked system requires reconfiguration to a desired formation with considering the constraints of neighborhood (local) interactions and limited resources. In this paper, a graph optimization problem presented that through solving it, the inter-agent correspondences determined. The optimization problem considers maximizing the graph connectivity through modification on edge connections.

Formation is a natural behavior. Birds and fish usually have collective motion in group foraging. They employ special geometrical patterns and as need due the environmental conditions, the reconfiguration in formation can be accomplished. Inspiring formation in nature and due its advantages for animal groups, engineers aim to employ formation in mobile agents. Formation control is a basic problem in multi-agent autonomous systems for applications in unmanned vehicle systems. Establishing formation of Unmanned Ground Vehicles [1], [2], Unmanned Aerial Vehicles (UAVs) [3], and Unmanned Underwater Vehicles [4] are some of formation applications inspired from quadrupeds, birds, and fish.

Dynamics of group interactions in a networked multi-agent system can be modeled through graph theory. The network topology sensitivity can be analyzed through algebraic connectivity. A network with maximized algebraic connectivity has better convergence speed, robustness and synchronization [5]. *Fiedler* eigenvalue [6] is prominent parameter to evaluate the connectivity properties in networked systems. A well connected network has better speed of convergence and its Fiedler eigenvalue has larger magnitude, i.e. well connectivity means the larger Fiedler eigenvalue and vice versa. In unweighted edge graphs, a main approach to maximizing connectivity or Fiedler eigenvalue is the modifying of the edge connections [7, 8]. Design of network topology considering connectivity, robustness and convergence is investigated in [7, 9]. Fiedler eigenvalue as convergence speed parameter is studied at [10, 11]. In this paper, the graph connectivity through maximizing Fiedler eigenvalue is the approach follow to design interaction network topology in desired formation. The aim is to reach better convergence speed when the agents move to perform reconfiguration.

A control strategy is needed to perform collective motion toward the desired formation after inter-agent correspondence is determined. This paper discusses about agents with double-integrator dynamics. The dynamics of each agent is coupled with its neighbors in the assumed multi-agent system. In such control problem, differential game theory and Nash equilibrium concept can be used to find optimal control input to agents. Game theory refers to mathematically studying of interactive decision making among a multi-agent team [12]. Formation control can be stated as a non-cooperative game [13], such that the team objective is to keep a formation while each individual involved pursues his/her own interest which is partly conflicting with others. The non-cooperative game solution is a Nash equilibrium concept that can be used as the formation strategy by the agents. Formation control problem was formulated as a linear-quadratic Nash differential game in [13]. Formation control of multiple UAVs under distributed information assumption was formulated as Nash differential game in [3]. Time-varying formation control using feedback information Nash differential game is investigated at [14].

This paper is organized as follows: system dynamics, graph theory basis, proposed graph optimization problem and open-loop Nash equilibrium are stated in section 2. A case study of formation reconfiguration for a four agents is derived in Section 3. Conclusion is stated in Section 4.

2. Problem Definition

2.1. System Dynamics

Suppose a multi-agent autonomous system with N agents. An agent i has a double integrator dynamics $(i:1...N)$

$$\dot{z}_i = az_i + bu_i \,. \tag{1}$$

where $z_i = \left[q_i^T, \dot{q}_i^T\right]^T \in R^{2n}$ and $u_i \in R^n$ is the state and control input vectors, respectively. Matrices a and b defined as

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes I_n \,,\; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes I_n \,.$$

where I_n is identity matric with order n and operator $\otimes$ shows the Kronecker product. Dynamics (1) is controllable and observable.

Let to represent all the agent states of the whole system with $z = [z_1,..., z_N]^T \in R^{2nN}$. Therefore, the whole system dynamics can be arranged as

$$\dot{z} = Az + \sum_{i=1}^{N} B_i u_i \,. \tag{2}$$

where matrices A and B_i are

$$A = I_N \otimes a \,,\; B_i = [0,...,1,...,0]^T \otimes b \,.$$

Also, let to define the state vector of a neighbor $j \in N_i$ to agent i as $z_{N_i}^j = \left[q_j^T, \dot{q}_j^T\right]^T \in R^{2n}$. Then, all the neighbors for an agent i can be shown as $z_{N_i} = \left[z_{N_i}^1,..., z_{N_i}^j,..., z_{N_i}^{N_i}\right]^T \in R^{2nN_i}$.

2.2. Graph Representations

Graph representation can be used to describe formation topology where a vertex in the graph shows an agent in the formation and an edge represents interaction between two agents. Let define a vertex and an edge set as $V = \{v_1,..., v_N\}$ and $E = \{(v_i, v_j) \in V^2\}$, respectively for the graph $G = (V, E)$ correspond with the system interaction topology. Graph G is assumed undirected, connected and has no multiple edges. Graph connectivity is necessary to keep formation control. Fig. 1 shows an undirected graph representation for an example with four agents.

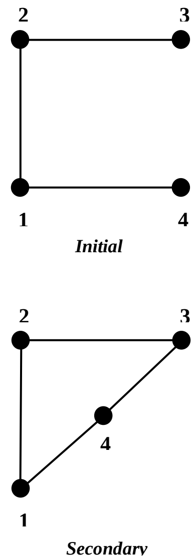


Fig. 1. Undirected graph representation for a reconfiguration problem in multi-agent formation.

For the undirected graph G, the *Laplacian* matrix is defined

$$L = D - A \,. \tag{3}$$

where $A = \lfloor a_{ij} \rfloor$ and $D = \lfloor \delta_{ij} \rfloor$, $(A, D \in \mathrm{R}^N)$ such that $a_{ij} = 1$ if and only if $(i, j) \in E$, $\delta_{ii} = \sum_j a_{ij}$. The Laplacian L is symmetric, positive semi-definite. All of its eigenvalues are real that can be ordered as $\lambda_1 \le \lambda_2 \le ... \le \lambda_N$.

In a formation, each agent should keep a given distance with its neighbors as well as network connectivity is hold. In n-dimension coordinate, desired distance between two nodes j and i is defined as $d_{ij}^d = \left|z_j^d - z_i^d\right|$. The formation error between two connected neighbors j and i can be defined as

$$\left|z_j - z_i - d_{ij}^d\right| . \tag{4}$$

Whole system formation error can be stated as *theorem* 1 ([15]).

Theorem 1. Consider a group of N mobile agents with double integrator dynamics (1). Whole group formation error can be stated as

$$\sum_{(i,j)\in E} \left(\left(z_i - z_i^d\right) - \left(z_j - z_j^d\right)\right)^2 = 2\left(z - z^d\right)^T M \left(z - z^d\right). \tag{5}$$

where $M = L \otimes I_{2n}$ and L is Laplacian matric.

Proof. Based on Laplacian definition (3), an entry of $L = \lfloor l_{ij} \rfloor \in R^N$ is

$$l_{ij} = \begin{cases} \delta_{ii} & \text{if } j = i \\ -a_{ij} & \text{if } (j,i) \in E, j \neq i \\ 0 & \text{if } (j,i) \notin E, j \neq i \end{cases}. \quad (6)$$

Let to define vectors $z_i - z_i^d = e_i$, $z_j - z_j^d = e_j$, and in global form $z - z^d = e$.

$$\begin{aligned} &\sum_{(i,j)\in E} \left(\left(z_i - z_i^d\right) - \left(z_j - z_j^d\right)\right)^2 = \sum_{i,j=1}^{N} a_{ij}\left(e_i - e_j\right)^2 = \\ &\sum_{i,j=1}^{N} a_{ij}\left(e_i\left(e_i - e_j\right) + e_j\left(e_j - e_i\right)\right) = \\ &\sum_{i,j=1}^{N} a_{ij} e_i \left(e_i - e_j\right) + \sum_{i,j=1}^{N} a_{ij} e_j \left(e_j - e_i\right) \end{aligned}. \quad (7)$$

Above relation can be rearranged as

$$\begin{aligned} &\sum_{i,j=1}^{N} a_{ij} e_i \left(e_i - e_j\right) = \sum_{i,j=1}^{N} a_{ij} e_i^2 - \sum_{i,j=1}^{N} a_{ij} e_i e_j = \\ &\sum_{i,j=1}^{N} a_{ij} e_i^2 - \sum_{i=1}^{N}\left(\sum_{j=1}^{N} a_{ij} e_j\right) e_i = \sum_{i,j=1}^{N} a_{ij} e_i^2 - \sum_{i=1}^{N}\left(\delta_{ii} e_j\right) e_i = \\ &\sum_{i,j=1}^{N} l_{ij} e_i e_j = e^T L e = \left(z - z^d\right)^T L \left(z - z^d\right) \end{aligned}. \quad (8)$$

By repeating again the above operation,

$$\sum_{i,j=1}^{N} a_{ij} e_j \left(e_j - e_i\right) = e^T L e = \left(z - z^d\right)^T L \left(z - z^d\right). \quad (9)$$

Finally,

$$\begin{aligned} &\sum_{(i,j)\in E} \left(\left(z_i - z_i^d\right) - \left(z_j - z_j^d\right)\right)^2 = \sum_{i,j=1}^{N} a_{ij}\left(e_i - e_j\right)^2 = \\ &2e^T L e = 2\left(z - z^d\right)^T L \left(z - z^d\right) \end{aligned}. \quad (10)$$

Since $z \in R^{2nN}$ while $L \in R^N$ then we have to resize the Laplacian matrix by multiplying to an identity matrix with proper dimension. This is shown as $M = L \otimes I_{2n}$ where $M \in R^{2nN}$.

2.3. Topological Optimization Problem

Assume $G^{\text{int}} = \left(V, E^{\text{int}}\right)$ is related with initial topology in the formation where V shows node set and E^{int} is the edge set. In the desired formation topology, graph $G^{\text{sec}} = \left(V, E^{\text{sec}}\right)$ is desired as follow; the graph G^{sec} should be simple, no multiple edge and connected, while it has maximum algebraic connectivity. Maximum edge number at G^{sec} is defined as $\left|E^{\text{sec}}\right| = K$. Therefore, the following optimization problem to maximize algebraic connectivity in the desired formation is proposed in this paper

$$\begin{aligned} &\max \lambda_2(L) \\ &\text{s.t. } \max\left|E^{\text{sec}}\right| = K \end{aligned}. \quad (11)$$

Since $\lambda_2(L)$ is used in computing $\lambda_2\left(G^{\text{sec}}\right)$, then they can used equivalently in analyzing of λ_2.

For a graph G with $\left|E^{\text{sec}}\right|$ edges and N nodes, the Laplacian matric L also can be factorized as

$$L = \Delta^T \Delta = \sum_{i=1}^{\left|E^{\text{sec}}\right|} \Delta_i \Delta_i^T. \quad (12)$$

where Δ_i is a column vector of the incidence matric $\Delta = \left[\Delta_1, \ldots, \Delta_{\left|E^{\text{sec}}\right|}\right]$. Due to (12), the optimization problem (11) is equivalent with following statement

$$\begin{aligned} &\max \lambda_2(L) \\ &\text{s.t. } L = \sum_{i=1}^{\left|E^{\text{sec}}\right|} \Delta_i \Delta_i^T \\ &\qquad \max\left|E^{\text{sec}}\right| = K \end{aligned}. \quad (13)$$

2.4. Connection Modification

When agent correspondence in desired formation is determined, agents should move to accomplish reconfiguration. Changing in agent correspondence should be start in this time, such that for each agent maybe some of its current connections will be leaved and some new will be appeared. This process is well to be continuously progressed during whole time interval. Inspired by continuous perceptron theory in neural networks, we proposed sigmoidal functions (14) and (15) to carrying the modify operation for agent correspondences.

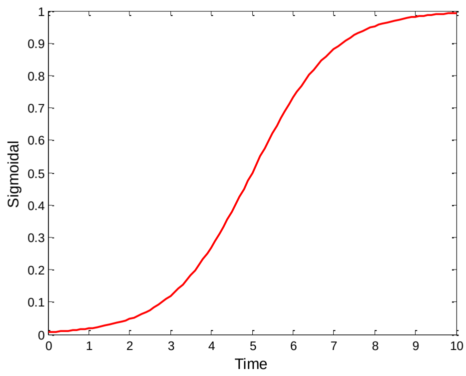
$$f(t, \lambda, c) = \frac{1}{1 + e^{-\lambda(t-c)}}. \quad (14)$$

$$g(t, \lambda, c) = 1 - \frac{1}{1 + e^{-\lambda(t-c)}}. \quad (15)$$

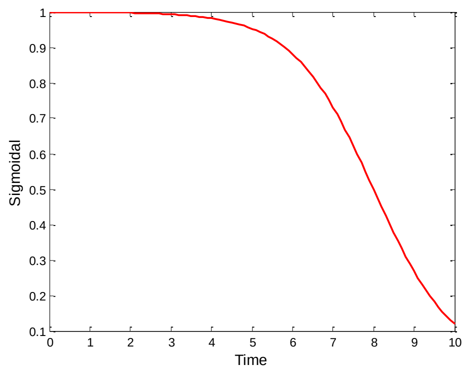
where $\lambda > 0$ is proportional to determining the steepness of the functions and constant c is a parameter to move the fire point in the functions to left or right side in the time interval. In other words, the sigmoidal functions do mapping on the time step vector t belong to a time interval $[0, T]$ depending on parameters of the steepness λ and constant c. In Fig. 2, (14) and (15) are shown for $T = 10$, $\lambda = 1$ and $c = 5$.

2.5. Nash Differential Games

The multi-agent system dynamics (2) subject to a given initial state z_0 is also can be considered as the state representation of a differential game. A finite horizon linear-quadratic cost function for the each agent/player can be considered. A Nash equilibrium can be obtained as bellow.



(a)



(b)

Fig. 2. (a) Sigmoidal curve for add a connection, (b) Sigmoidal curve for leave a connection.

Theorem 2. Consider a game with N players under state dynamics (2) and finite horizon linear-quadratic cost function for each player. The differential game has a unique linear feedback Nash equilibrium if and only if there exist a set of symmetric solution K_i to the coupled differential equations

$$\begin{aligned} \dot{K}_i(t) &= -A^T K_i(t) - K_i(t)A - Q_i(t) + K_i(t)\sum_{j=1}^{N} S_j K_j(t) \\ K_i(T) &= Q_{if} \end{aligned} \quad . \quad (16)$$

where $S_i = B_i R_{ii}^{-1} B_i^T$. Then unique open-loop Nash equilibrium for each initial state can be obtained as

$$u_i(t) = -R_{ii}^{-1} B_i^T K_i(t)\phi(t,0)z(0) . \quad (17)$$

$$\dot{\phi}(t,0) = \left(A - \sum_{j=1}^{N} S_j K_j(t) \right)\phi(t,0) . \quad (18)$$

$$\phi(0,0) = I . \quad (19)$$

Proof. The proof is given in [16].

$Q_{if} = M$, $Q_i = M$ and $R_{ij} > 0$ is weight matrix for control input. Q_{if} , Q_i are symmetric and positive semi-definite.

3. Case Study

A case study with four agents is presented. Graph information for the initial and desired formations are given as, V=4, E^{int}=3, K=4. The agents' positions in initial and desired formations in two-dimensional coordinate are given as z= [2 2; 2 4; 4 4; 4 2]^T and z^d= [10 8; 10 10; 11.41 9.41; 12 8]^T vectors, respectively. The agents 1, 2, 3 and 4 in initial formation are neighbor with {2, 4}, {3, 1}, {2} and {1}, respectively while in second formation, each agents is neighbor with all the rest.

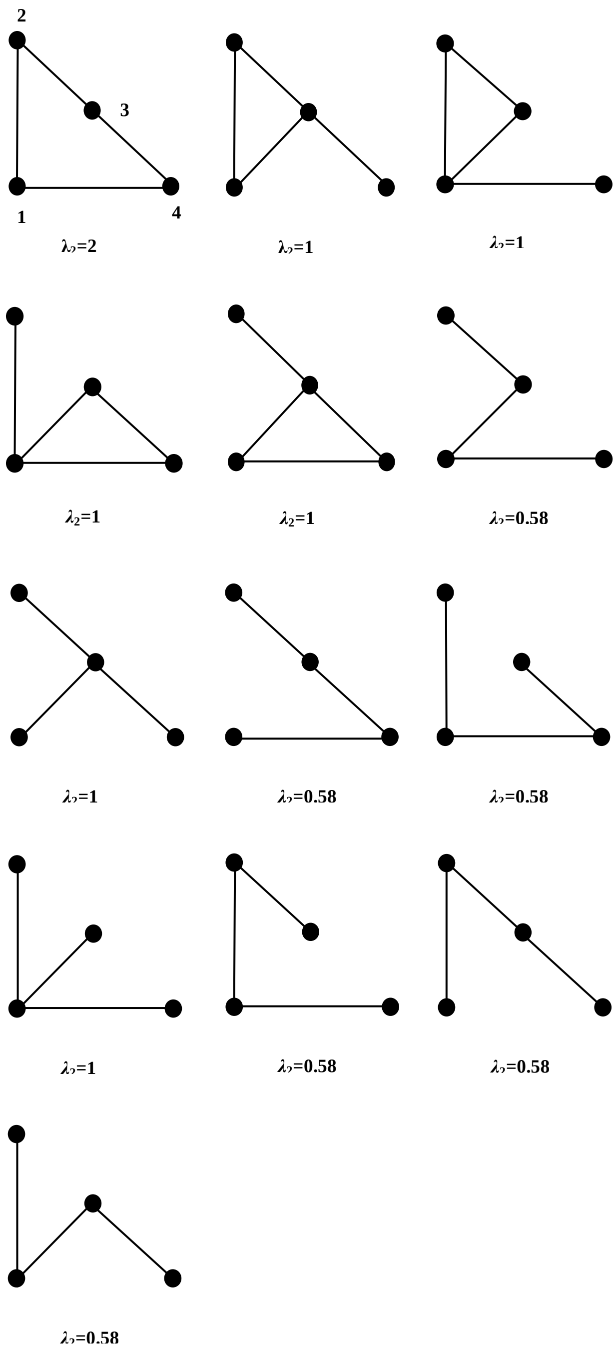


Fig. 3. Different graph (interaction network) topologies with corresponding Fiedler eigenvalue.

In the desired formation, agent 3 placed closer to the rest agents and then it can sense all the rest agents. The system can move to the desired formation with different interaction topologies as shown at Fig. 3. In this figure, Fielder eigenvalue (algebraic connectivity parameter) of each network's topology is shown, also. Since the desired formation with maximum algebraic connectivity is well, then from Fig. 3, it can be seen which topology is most desirable.

Because of small agent society size in this case study, possible graph topologies and selection of well topology was obtained simply through a manually solution to the optimization problem (13). In general, an algorithm is needed to solve (13). In [10], a greedy perturbation heuristic algorithm is proposed to solving the problem of adding edges from a set of candidate edge to a graph aiming to maximize the Fielder eigenvalue. We used this algorithm as a basis and proposed an algorithm as shown in Fig. 4 to solve (13). The algorithm uses the edge (i, j) which cause the largest value of $(v_i - v_j)^2$, where v is a Fiedler eigenvector of present Laplacian matrix.

When agent correspondence is determined for desired formation, each agent Nash control strategy can be obtained by solutions presented in theorem 2. Also, the process of changing agent correspondence can be described continuously in time by using following incidence matrix,

$D(t)$= [-1 0 -1 0; 1 -1 0 0; 0 1 0 $g(.)$; 0 0 1 $f(.)$]

Based on theorem 1 the formation error is directly related to Laplacian matrix. We use $M(z - z^d)$ to compute formation error when the agents move to desired formation. As shown in Fig. 5, formation error magnitude using the topology with $\lambda_2 = 2$ is less than the case topology of $\lambda_2 = 0.58$ is used. This means that less control effort is needed to accomplish physically move to perform reconfiguration using the topology with largest λ_2.

READ *NodeSet*
READ each *Node-position*
READ each *Node-neighbors*
BUILD a *base-graph* is connected with minimal edge
connections subject to each *Node-neighbors*
COMPUTE Laplacian *L*, second smallest eigenvalue
λ_2(L) and its eigenvector *v*
REPEAT
ADD an edge (i, j) with largest $(v_i - v_j)^2$ where *i*
and *j* are in each other *Node-neighbors*
UNTIL maximum edge number reaches *K*
DISPLAY λ_2(L)

Fig. 4. Algorithm proposed to solve the optimization problem.

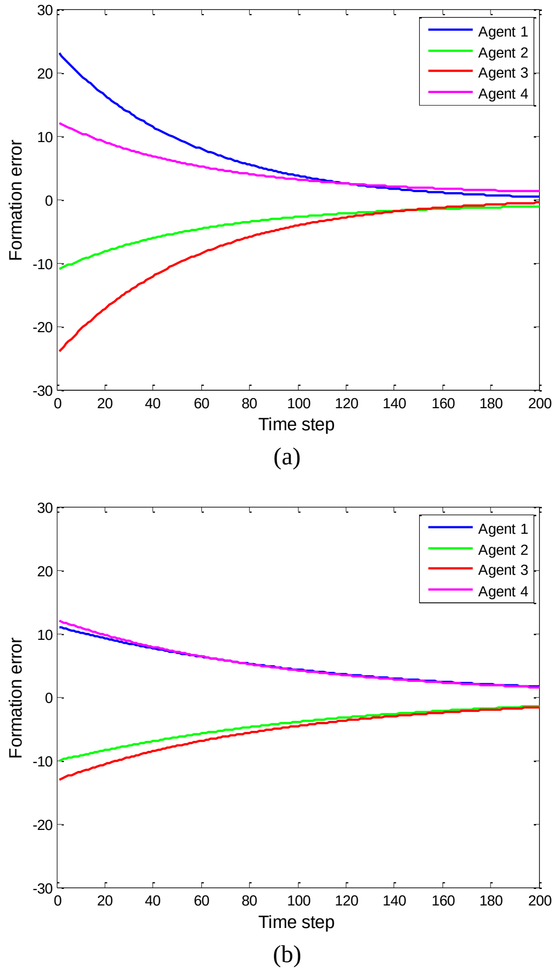


Fig. 5. (a) Formation error with λ_2= 0.58, (b) Formation error with λ_2= 2.

4. Conclusions

Algebraic connectivity is used as a parameter to design well-structured formation topologies. An optimization problem is proposed to obtain topologies with maximum Fiedler eigenvalue. An algorithm is presented to solve the problem. Two sigmoidal functions are introduced to use in the incidence matrix of desired formation topology to make changes in the agent correspondence continuously in time. A case study is derived to measure formation error of a team with four agents when the different topologies are used. Results showed that the formation error is less when the topology has well connectivity and therefore minimum control effort is needed to accomplish reconfiguration.

5. References

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