Some Statistical Characteristics for the APD

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***Abstract* - The Webb, McIntyre, Conradi (WMC) distribution has been often used to approximate the APD (avalanche photodiode) receiver output statistics. In this paper we present some interesting properties of WMC distribution which shed new light on the subject .A recursion relation for calculating the probability distribution for gains in physical avalanche diode is derived.**

1. INTRODUCTION

The introduction of the WMC distribution greatly simplifies the APD model. Recently it has been shown, [9], [10], that the WMC distribution and inverse Gaussian distribution are of the same type. By using this fact Duma[2] (1996) and Tang and Letaief [1] (1998) have obtained expressions for the moment generating function (MGF) and cumulant generating function (CGF) of WMC distribution, a key element which is needed in various BER (bit-error-rate) evaluation methods [1] and to estimate the sensitivity of backscattering technique [2].This paper presents new properties of the WMC distribution which allows certain comparison with normal distribution and exact distribution for secondary electrons generated in APD.

2. APD MODEL

The avalanche detector is a device in which thermally or optically generated hole-electron pairs generate additional hole-electron pairs through collision ionizations. This statistical process is called avalanche multiplication.

Consider the probabilities P(m/n) that n initial carriers will result in a total of m pairs. P(m/n) had been originally derived from special cases by McIntyre [4] and verified experimentally by Conradi , and was rigorously proven by P.Balaban, P.E.Fleischer, and H.Zucker [8].

Personick [3] has shown that the moment generating function

$$M(e^s) = \sum_{m=1}^{\infty} P(m/1)e^{sm} \qquad (1)$$

of the gain g of the diode is given implicitly by

$$s = \ln M - \frac{1}{1-k}\ln[(1-a)M+a] \qquad (2)$$

where

$$a = \frac{1+k(G-1)}{G}$$

in terms of the average gain of the diode $G=E(g)$ and k- the ionization ratio of holes to electrons.

$P(m/n)$ is the n-fold convolution of $P(m/1)$ with itself:

$$P(m/n)=P^{*n}(m/1). \qquad (3)$$

Let $M_n(e^s)$ be the moment generating function of $P(m/n)$:

$$M_n(e^s) = \sum_{m=0}^{\infty} P(m/n)e^{sm} \qquad (4)$$

Using (3) one can write $M_n(z)=M^n(z)$ where $z=e^s$.

From (2) and (4) Balaban, Fleischer and Zucker [8] obtain $P(m/n)$:

$$P(m/n) = \frac{n[\frac{(1-k)(G-1)}{G}]^{m-n}\Gamma(\frac{m}{1-k})[\frac{1+k(G-1)}{G}]^{c}}{(1-k)c(m-n)!\Gamma(c)} \qquad (5)$$

where: $c = \dfrac{n+k(m-n)}{1-k}$.The special case of the $n=1$, which describes the gain g distribution has been derived by Mazo and Salz [12].

Consider a point process representing the primary (photon-generated) carriers. Let the number of these carriers generated within the time interval $[0, \Delta t)$ be described by the discrete random variable $N_{\Delta t}$. Let $q_n = Pr(N_{\Delta t} = n)$. If the detector is illuminated by incident power $p(t)$ then the average number of electron-hole pairs $\bar{n}$ generated in time Δt is

$$\bar{n} = \frac{\eta}{hf}p(t)\Delta t + \lambda_0 \Delta t$$

where λ_0 is the dark current in number of pairs /sec., f is the optical frequency, η is the quantum efficiency and h is Planck's constant.

The electron count m generated over the Δt second observation time is a random variable governed by the randomness of the field photodetection and avalanche amplification. The probability distribution of m electrons occurring after avalanche multiplication is given by

$$P(m) = \sum_{n=0}^{\infty} q_n P(m/n). \qquad (6)$$

If the actual number of electron-hole pairs generated in time Δt is a Poisson process, where

$$q_n = \frac{(\bar{n})^n e^{-\bar{n}}}{n!}$$

the random variable m is characterized by the Conradi distribution

$$P(m) = \sum_{n=1}^{n} \frac{e^{-\bar{n}}(\bar{n})^n}{n!} P(m/n). \qquad (7)$$

Standard system analysis is complicated by the complexity of the counting model in (2). A useful approximation is given by Webb, McIntyre, and Conradi [7] as

$$P(m/\bar{n}) = \frac{1}{\sqrt{2\pi\sigma}} \frac{1}{\left(1+\frac{m-\bar{m}}{\sigma\lambda}\right)^{\frac{3}{2}}} \exp\left\{-\frac{(m-\bar{m})^2}{2\sigma^2(1+\frac{m-\bar{m}}{\sigma\lambda})}\right\} \qquad (8)$$

where k=effective ionization ratio, G=average avalanche gain, $\bar{m} = \bar{n}G$, $\sigma^2 = \bar{n}G^2 F$,

$$F = kG + [(2 - G^{-1})(1-k)]$$

$$\lambda = (\bar{n}F)^{\frac{1}{2}} / (F-1).$$

For the APD model the current $i(t)$ is

$$i(t) = \frac{me}{\Delta t}$$

where e is the charge of an electron. The WMC probability density function for the output current of diode $i(t)$ is (9):

$$p(y) = \frac{1}{\sqrt{2\pi}\sigma_i} \frac{1}{\left(1+\frac{y-M}{\sigma_i\lambda}\right)^{\frac{3}{2}}} \exp\left\{-\frac{(y-M)^2}{2\sigma_i^2\left(1+\frac{y-M}{\sigma_i\lambda}\right)}\right\}$$

where $\sigma_i = \sigma\frac{e}{\Delta t}$, $M = \frac{\bar{n}e}{\Delta t}G$ =mean output current.

A. Avalanche Gain Statistics

Taking the derivative in (2) we obtain(10)

$$\frac{dM}{ds}[a(1-k) - k(1-a)M] = M^2(1-k)(1-a) + a(1-k)M]$$

Note that (10) is singular when

$$M(s_c) = \frac{a}{1-a}\frac{1-k}{k}$$

beyond which $M(s)$ does not exists.Clearly, for $s>s_c$, $dM/ds<0$, which leads to a contradiction. Notice also that $dM/ds\uparrow\infty$ as $s\uparrow s_c$ ($\ln M$ is said to be steep).

Using (10) after multiplication by M^{n-1} and integration we find:

$$(e/n)M^n(s) - (b/(n+1))M^{n+1}(s) = r\int MM^n ds + e\int M^n ds \qquad (11)$$

with $r=(1-a)(1-k)$, $b=(1-a)k$ and $e=a(1-k)$.

Substituing $M_e(e^s)$, given by (1) and $M^n=M_n$ given in (4), in (11) it is easy to obtain the following recurrence relations (rellabelling $P(m/n)= P_{m,n}$ and $P(m/1)=p_m$):

$$P_{i,n} = \frac{bi + r(n-1)}{(i-n)\cdot(n+1)\cdot e} \cdot \sum_{j=1}^{i-n} p_j \cdot P_{i-j,n} \qquad (12)$$

In the particular case n=1 one obtains

$$p_{2l} = ((2l-1)e)^{-1}(2r+2lb)(p_l^2/2 + p_1 p_{2l-1} + \ldots\ldots + p_{l-1}p_{l-1})$$
$$(12')$$
$$p_{2l+1} = (2le)^{-1}((2l+1)b+2r)(p_1 p_{2l} + p_2 p_{2l-1} + \ldots + p_l p_{l-1})$$

From (5) one easily obtains $P_{n,n}=a^{n/(1-k)}$. We can find $P_{n,n}$ in another way. We write (2) as follows:

$$e^{s\cdot n} = \frac{M^n}{((1-a)\cdot M + a)^{\frac{n}{1-k}}} = \frac{P_{n,n}\cdot e^{sn} + P_{n-1,n}\cdot e^{s(n-1)} + \ldots}{((1-a)\cdot M + a)^{\frac{n}{1-k}}}$$

If $s\to -\infty$, we obtain $P_{n,n}=a^{n/(1-k)}$; $M(s)\to 0$ when $s\to -\infty$.

Similarly, an equivalent form of (12) result:

$$P_{i,n} = \frac{n}{e\cdot(i-n)} \cdot \sum_{j=1}^{i-n} (bj + r)\cdot p_j \cdot P_{i-j,n} \qquad (12')$$

This result is consistent with (5). For the particular case n=1 the equivalence was demonstrated by D-R. Popescu of Bucharest University [14]. We hope that these results can reduce the computing time and improve the accuracy.

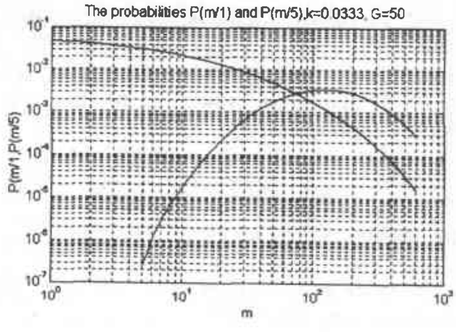

Fig. 1 The probability distribution of APD gain whwn G is 50 and k=0.0333..

B. Cumulants of P(m)-Conradi distribution.

Let g_k be gain for the *k*-th incoming electron, *n* the Poison distributed number of primary electron emitted in an interval Δt and *m* the number of electrons after multiplication

$$m=\sum_{1}^{n} g_k$$

Let now $M(e^s)$ be the generating function of $P(m/1)$, $N(e^s)$ that of the Poisson distribution and $Q(e^s)$ for P(m). We define the cumulant generating functions $K_M(s)=\ln M(e^s)$, $K_N(s)=\ln N(e^s)$ and $K_Q(s)=\ln Q(e^s)$.

It can be shown that $K_Q(s)=K_N(K_M(s))$. Since $K_N(s)=\bar{n}(e^s-1)$ one obtains:

$$K_Q(s)=\bar{n}(M(e^s)-1).$$

This implies

$$\left.\frac{d^i K_Q(s)}{ds^i}\right|_{s=0}=\bar{n}\left.\frac{d^i M(s)}{ds^i}\right|_{s=0}$$

Thus the cumulants of *P*(*m*) are $\bar{n}E(g^j)$ where $E(g^j)$ are the moments of *g*. These can be calculated from the following recursive relations

$$E(g^l)=\sum_{j=1}^{2l-1} A_{j,l}G^j$$

$V_l=[A_{1,l}, A_{2,l},\ldots, A_{2l-1,l}]$

$V_{l-1}=V_lA$

where

$$A=\begin{Bmatrix} a_1 & a_2 & a_3 & 0 & \ldots & 0 & 0\\ 0 & 2a_1 & 2a_2-1 & 2a_3 & \ldots & 0 & 0\\ 0 & 0 & 3a_1 & 3a_2-2 & \ldots & 0 & 0\\ 0 & 0 & 0 & 4a_1 & \ldots & (2l-2)a_3 & 0\\ 0 & 0 & 0 & 0 & \ldots & (2l-1)a_2-2l+2 & (2l-1)a_3 \end{Bmatrix}$$

and $a_1=k-1$, $a_2=2(1-k)$, $a_3=k$.

3. WMC DISTRIBUTIONS

We seek the statistics of the random total number *m* of hole-electron pairs which result ultimately through collision ionization's as given by (3), and of the output current *i(t)* with pdf given by (4). In (9) it is useful to make the substitution

$$U=1+\frac{Y-M}{\sigma\lambda}$$

or, equivalently

$$Y=\sigma\lambda U+M-\sigma\lambda\equiv AU+B \qquad (13)$$

One obtains the probability density function for the random variable *U*

$$p(u)=\frac{\lambda}{\sqrt{2\pi u^3}}\exp\left\{-\frac{(u-1)^2\lambda^2}{2u}\right\}$$

i.e. an inverse Gaussian(Wald) distribution with parameters $b=\lambda^2$, a=1[11] [9].

In[9] the cumulant generating function (14)

$$\Psi_Y(t)=iBt+\Psi_U(At)=iBt+b\left\{1-\left(1-\frac{2iAt}{b}\right)^{\frac{1}{2}}\right\}$$

is obtained. The first four cumulants corresponding to random variable Y are

$k_1=A+B=M=$ mean output current;

$k_2=A^2/b=\sigma^2$ = variance of the diode current; $k_3=3A^3/b^2=3\sigma^3\lambda^{-1}$, $k_4=15\sigma^4\lambda^{-2}$, and generally, when $r\geq 2$

$$k_r=1\cdot3\cdot5\cdot\ldots\cdot(2r-3)\sigma^r\lambda^{2-r} \qquad (15)$$

The first two cumulants of the Conradi distribution and for the WMC distribution are identical but the cumulants of order *n* >3 are entirely different

For the random variable *Y* the central moments μ_r can be obtained by a recurrence relation starting from the cumulants k_r that is

$$\mu_{r+1}=k_{r+1}+\sum_{j=1}^{r-1}C_r^j k_{j+1}\mu_{r-j}\ ,\ r\geq 2 \qquad (16)$$

with $\mu_1=0$ and $\mu_2=k_2$.

Using (14) the following property may be shown

P₁. When $Y_1, Y_2, \ldots, Y_n$ are independent random variables and Y_i is distributed according to (9) with parameters M_i, σ_i, λ_i then the distribution of $Z=d\sum_{i=1}^{n} Y_i$ is also of form (9) with

$$M=d\sum_{i=1}^{n} M_i\ ,\ \sigma^2=d^2\sum_{i=1}^{n}\sigma_i^2\ ,\ \lambda^2=\sum_{i=1}^{n}\lambda_i^2$$

This follows from (12) if we note that

$$\frac{A}{b}=\frac{\sigma\lambda}{\lambda^2}=\frac{\sigma}{\lambda}=\text{constant}.$$

(for a given APD). If $d=1/n$ we obtain: the arithmetic mean of *n* independent, identically distributed random WMC variables with pdf (9) has a WMC pdf with parameters:

$$M_1=M,\ \sigma_1^2=\sigma^2/n,\ \lambda_1^2=n\lambda^2 \qquad (17)$$

We can see that M, σ^2, λ^2 are proportional to the average photogenerated charge per pulse when Y_i represents the current of the diode in the *i*-th Δt interval.

The assumption that the statistical distribution of the output voltage pulse amplitude can also be represented by the statistical distribution of the integrated charge per pulse from the detector itself (the first central assumption of Conradi in [6]) is strictly true only for integrate and dump receiver.

P2. Let g_e be the effective gain of a APD $g_e = m/\overline{n}$. The distribution of the random variable g_e is WMC. The cumulant generating function may be calculated to give

$$\Psi_{g_e}(t) = -\frac{G}{F-1}it + \frac{\overline{n}F}{(F-1)^2}\left\{1-\left[1-2it\frac{G(F-1)}{\overline{n}}\right]^{\frac{1}{2}}\right\} \tag{18}$$

If $\overline{n} = 1$ in (18), then one obtains the expression (5) in [7] derived by House in his dissertation (see also [13]).

P3. Using (10) and (11) one obtains the cumulative distribution function of a WMC distribution (16)

$$F_Y(y) = G\left(\frac{y-A-B}{\sqrt{A(y-B)}}\cdot\lambda\right) + e^{2\lambda^2}\cdot G\left(-\lambda\cdot\frac{y+A-B}{\sqrt{A(y-B)}}\right)$$

where $\lambda = \sqrt{\overline{n}F}/(F-1)$, $A = MF/(F-1)$, $B = -M/(F-1)$ and $G(\cdot)$ is the standard gaussian distribution function.

P4. If Y is WMC distributed with parameters M, σ, λ then as $\sigma/\lambda \equiv G(F-1)$ approaches 0, the distribution of Y becomes normal with mean M and variance σ^2.

Proof:

$$\lim_{F\to 1} \Psi_Y(t) = \lim_{\frac{a}{b}\to 0}\left(iBt - b\left(1 - 1 - \frac{Ati}{b} - \frac{A^2t^2}{2b^2} - \ldots\right)\right)$$

$$\lim_{F\to 1} \Psi_Y(t) = iMt - \frac{\sigma^2 t^2}{2}$$

P5. WMC distribution function is an infinitely divisible distribution.

P6. The saddlepoint approximation for the pdf for the arithmetic mean of n independent, identically distributed random WMC variables, is also WMC pdf with the parameters given by (17).

4. CONCLUSIONS

A number of problems are considerate relevant to understanding of the exact distribution and an approximation of the APD statistics which is called the WMC distribution. In this paper we derived a recurrence relation for calculating the probability distribution for APD gain and list some interesting properties of WMC distribution and exact distribution.

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