PID Controller Design Based on Second Order Model Approximation by Using Stability Boundary Locus Fitting

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Abstract

This study presents a model reduction method based on stability boundary locus (SBL) fitting for PID controller design problems. SBL analysis was commonly applied for controller stabilization problems. However, we use SBL analysis for the reduction of high order linear time invariant system models to second-order approximate models to facilitate analytical design of closed loop PID control systems. The PID design is implemented by a multiple pole placement strategy which enforces the control system had real poles with a desired time constant specification. Illustrative design examples are presented for the analytical PID design of high-order plant models by means of second-order SBL model approximations.

1. Introduction

Effective model order reduction strategies are mainly needed for the development of analytical methods for the solutions of design and analysis problems of control engineering practice due to fact that real control problems may introduce high-order models. Model order reduction aims to obtain a low-order approximation of high-order systems which exhibits similar time and frequency response as much as possible under the operation ranges of control systems. In the control points of view, it is very desirable to have a good approximation for time and frequency domain properties of high-order systems, particularly within the low frequency region. Some important studies on model order reduction can be found in [1-8].

In the current paper, we proposed a model identification approach that fits SBL of the high-order LTI model of original systems to SBL of a lower order approximate LTI model in the closed loop PI controller design plane. Here, we consider SBL lines of original system as the finger prints of original system model and hence SBL of the order reduced system model is fitted the sampled SBL points of the original LTI model. In the literature, SBL analyses were widely utilized for stabilization of PID control systems [9-11]. SBL analyses are based on the solution of characteristic equation of unity feedback closed loop control system for controller parameters. SBL is mainly plotted in (k_p , k_i) parameter plane to illustrate visually the stability region of PI controller coefficients. Essentially, the fitting of SBL lines of original model and approximate models provide an approximation of characteristic equations of their transfer functions at the sampled frequency points and the main advantage of SBL fitting in model approximation is that it aims to match stability regions of original and approximate models.

Also, in the paper, we employ the model order reduction based on SBL fitting in the analytical design problem of closed loop PID control systems. Firstly, we obtain the second order model reduction of high order plant function by the fitting the SBL in a predefined frequency range and express the coefficients of the second order approximate plant model by solving linear equation set. Secondly, we used these coefficients for design of PID controller that formulated based on pole placement on real axis for a desired time constant specification of control systems. These two tasks provide a straightforward solution for the analytical PID tuning problem for high-order plant models according to second-order approximate models. Since the second order models have a phase limit of -180^o, the proposed method is very effective for the high-order plants with a phase limitation of -180^o. Numerical examples illustrate application of the proposed method for analytical controller tuning and model identification problems.

2. Methodology

2.1. Theoretical Background

This section summarizes calculation of SBL for high order plant transfer functions according to closed loop PI control system given in Figure 1 [9,10]. SBL analysis was employed to figure out the ranges of controller parameters stabilizing control systems. In this study, we used SBL curvatures for model approximation proposes. By considering transfer function of PI controller as $C(s) = k_p + k_i s^{-1}$ and transfer function of the high-order plant model as $G(s) = N(s)/D(s)$ in Figure 1, one can write the characteristic equation of the system as follows,

$$\Delta(s) = 1 + (k_p + k_i s^{-1})G(s) = sD(s) + (k_p s + k_i)N(s) = 0 \qquad (1)$$

In order to obtain SBL of $G(s)$, characteristic equation given by equation (1) is solved for $s = j\omega$ in frequency domain. When this complex characteristic equation, expressed as $\Delta(j\omega) = 0$, is solved real and imaginary parts of the complex characteristic equation yield the following equation set:

$$k_i N_R(\omega) - k_p \omega N_I(\omega) = \omega D_I(\omega) \qquad (2)$$

$$k_i N_I(\omega) + k_p \omega N_R(\omega) = -\omega D_R(\omega) \qquad (3)$$

where, $N_R(\omega)$ and $N_I(\omega)$ are real and imaginary parts of the numerator polynomial. Similarly, the polynomials $D_R(\omega)$ and $D_I(\omega)$ are real and imaginary parts of denominator polynomial. By solving $k_p(\omega)$ and $k_i(\omega)$ from equations (2) and (3), we obtain SBL line denoted by $S(k_p(\omega), k_i(\omega))$ in (k_p , k_i) plane as follows.

$$k_p(\omega) = -\frac{N_I(\omega)D_I(\omega) + N_R(\omega)D_R(\omega)}{N_I(\omega)^2 + N_R(\omega)^2} \qquad (4)$$

$$k_i(\omega) = \frac{\omega N_R(\omega)D_I(\omega) - \omega N_I(\omega)D_R(\omega)}{N_I(\omega)^2 + N_R(\omega)^2} \qquad (5)$$

The SBL of plant $G(s)$ can be expressed as a set of points in (k_p, k_i) plane as

$$S = \{(k_p(\omega), k_i(\omega)) : \Delta(k_p, k_i, \omega) = 0 \wedge \omega \in (-\infty, \infty)\} \tag{6}$$

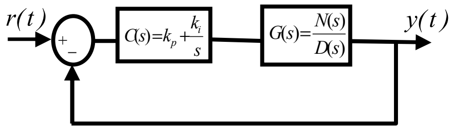

Fig. 1. The unity feedback PI control system for SBL calculation of high order plant model $G(s)$.

2.2. Second Order Model Reduction by SBL Fitting

Lets express the second-order approximate system in a general form as follows,

$$G_2(s) = \frac{a_o}{b_2 s^2 + b_1 s + b_0} \tag{7}$$

The characteristic polynomial of unity feedback closed loop PI control system for $G_2(s)$ can be expressed as,

$$\Delta_2(s) = 1 + C(s) G_2(s) \tag{8}$$

In order to find out SBL of $G_2(s)$ as explained in previous section, the characteristic equation of system is written in the frequency domain as,

$$\Delta_2(j\omega) = k_i a_0 - b_1 \omega^2 + j\omega(b_0 + k_p a_0 - b_2 \omega^2) = 0 \tag{9}$$

To find the coefficients of $G_2(s)$, which are fitting the sampled SBL points $\{(S(k_{p1}(\omega_1), k_{i1}(\omega_1))$, $S(k_{p2}(\omega_2), k_{i2}(\omega_2))\}$ for angular frequency sampling $\omega_i \in [\omega_{\min}, \omega_{\max}]$, one can write the following equation sets by considering equation (9) ,

$$\begin{aligned} \Delta_2(j\omega_1) &= k_i(\omega_1) a_0 - b_1 \omega_1^2 \\ &+ j\omega_1(b_0 + k_p(\omega_1) a_0 - b_2 \omega_1^2) = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} \Delta_2(j\omega_2) &= k_i(\omega_2) a_0 - b_1 \omega_2^2 \\ &+ j\omega_2(b_0 + k_p(\omega_2) a_0 - b_2 \omega_2^2) = 0 \end{aligned} \tag{11}$$

Then, four homogenous linear equations to obtain four plant function coefficients $\{a_0, b_0, b_1, b_2\}$ are written for two real part equations as,

$$k_i(\omega_1) a_0 - b_1 \omega_1^2 = 0 \quad \& \quad k_i(\omega_2) a_0 - b_1 \omega_2^2 = 0 \tag{12}$$

and two imaginary part equations as follows,

$$\omega_1 b_0 + k_p(\omega_1) \omega_1 a_0 - b_2 \omega_1^3 = 0$$

$$b_0 \omega_2 + k_p(\omega_2) a_0 \omega_2 - b_2 \omega_2^3 = 0 \tag{13}$$

The solution of these homogenous equations can be dependent on an arbitrary a_0 as follows,

$$b_0 = a_0 \left(\frac{k_p(\omega_2) - k_p(\omega_1)}{\omega_2^2 - \omega_1^2} \omega_1^2 - k_p(\omega_1) \right) \tag{14}$$

$$b_1 = a_0 k_i(\omega_1) / \omega_1^2 \tag{15}$$

$$b_2 = a_0 \frac{k_p(\omega_2) - k_p(\omega_1)}{\omega_2^2 - \omega_1^2} \tag{16}$$

When these equations are used in second-order approximate system model (equation (7)), a_0 coefficients can be reduced because of a_0 factorization in numerator and denominator polynomials and one obtains the reduced model with respect to sampled SBL points as,

$$G_2(s) = \frac{1}{\left(\frac{k_p(\omega_2) - k_p(\omega_1)}{\omega_2^2 - \omega_1^2}\right) s^2 + (k_i(\omega_1) / \omega_1^2) s + \left(\frac{k_p(\omega_2) - k_p(\omega_1)}{\omega_2^2 - \omega_1^2} \omega_1^2 - k_p(\omega_1) \right)} \tag{17}$$

According to equation (17), while calculating numerically b_0, b_1 and b_2 by equations 14-16, one can select value of the coefficient a_0 equal to one for the sake of simplicity. Here, SBL sampling points from high-order model $G(s)$, which are $k_p(\omega_1)$, $k_p(\omega_2)$, $k_i(\omega_1)$ and $k_i(\omega_2)$, can be calculated by considering equations (4) and (5) for a desired frequency sampling $\omega_i \in [\omega_{\min}, \omega_{\max}]$. The $G_2(j\omega)$ permits the second-order dynamics containing squared of frequency (ω^2) in frequency domain and provides SBL (equation (4) and (5)) written as,

$$k_p(\omega) = \frac{b_0 + b_2 \omega^2}{a_0} \text{ and } k_i(\omega) = \frac{b_1 \omega^2}{a_0} \tag{18}$$

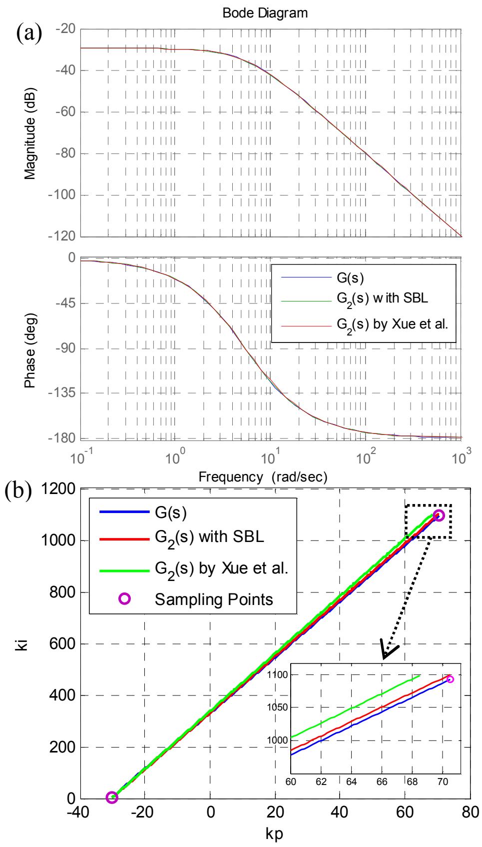

Fig. 2. The Bode diagrams and SBLs of the second order reduced models obtained by SBL model reduction method and Xue's optimal model reduction method.

We used the phase limitation of the second order transfer functions to determine whether the high-order model can be reduced to the second order model approximately. The phase of the second order model can be expressed as,

$$Arg(G_2(j\omega)) = -\tan^{-1}(b_1\omega/(b_0 - b_2\omega^2)) \quad (19)$$

The phase of $G_2(j\omega)$ is limited to $|Arg(G_2(j\omega))| < \pi$ for $0 < \omega < \infty$. When the phase of the high-order model is in the range of $|Arg(G(j\omega))| < \pi$, this system can be reduced to the second order approximate model for PID control system design. This condition is referred to phase limitation condition for second order model reduction.

Figure 2 shows the bode diagrams and SBL lines of the second order reduced models of a high-order plant model,

$$G(s) = \frac{s^4 + 10s^3 + 35s^2 + 50s + 24}{s^6 + 21s^5 + 174s^4 + 7351s^3 + 1624s^2 + 1764s + 720} \quad (20)$$

This plant transfer function complies with phase limitation condition of second order model reduction and second order approximate model well represents this plant model. Figure 2(a) demonstrates that frequency responses of the proposed SBL model reduction method and Xue's optimal model reduction method [7] are well agreement with the frequency response of high-order model. Figure 2(b) clearly shows that SBL model reduction method can provide better fitting to SBL of high order model. Because, SBL model reduction method provides a solution enforcing to match SBL lines at the sampling points.

2.3. Analytical PID Control System Design for Second Order Plant Models

In order to tune PID controller with respect to a desired time constant specification of the closed loop control system, we place all poles of the closed loop PID control system on the real axis for equal time constants.

The closed loop PID control system with the second-order plant model produces three system poles. Lets denote them by P_1, P_2 and P_3. Our pole placement constraint for the equal real poles is expressed as follows,

$$P_1 = P_2 = P_3 = -\frac{1}{\tau} \quad (21)$$

where, parameter τ is the dominating time constant of control system. For enforcement of the multiple pole placement constraint in PID design, the characteristic polynomial of control system established as follows,

$$\Delta(s) = \left(s + \frac{1}{\tau}\right)^3 \quad (22)$$

The characteristic polynomial of the closed loop PID control system for the second-order approximate model, defined by equation (7), is written as,

$$\Delta(s) = b_2 s^3 + (b_1 + a_0 k_d)s^2 + (b_0 + a_0 k_p)s + a_0 k_i \quad (23)$$

By equating equation (22) and (23), one obtains the PID coefficients regarding to the pole placement constraint given in equation (23) as,

$$k_p = \frac{3b_2/\tau^2 - b_0}{a_0}\ ;\ k_i = \frac{b_2/\tau^3}{a_0}\ ;\ k_d = \frac{3b_2/\tau - b_1}{a_0} \quad (24)$$

The main advantages of this PID design strategy can be summarized for the control of second-order plants as follows,

(i) It enables to design PID controller by single parameter specification (time constant specification) and it also ensures the stability of the control system via the real system pole placement.

(ii) Due to providing high gain control by setting a low time constant, the proposed PID design can exhibit good disturbance rejection performance for second-order plants.

Figure 3 shows PID design of the second order unstable plant $G_2(s) = 1/(s^2 - s + 1)$ for various time constant. As seen in figure, settling time of the second order system is about $T_s \cong 10\tau$. This enables to design PID controller with respect to settling time specification.

Figure 4 shows unit step and unit disturbance performance of these PID designs for $G_2(s) = 3/(s^2 + s + 3)$ [12]. Haeri method [11] provides a smooth step response with low disturbance rejection. The method of Shen et al. [12] provides faster step response with high overshoot. However, the proposed PID design method with $\tau = 0.2$ can provide better step response and disturbance rejection performance for the second order system because of the high gain control and negative reel pole placement.

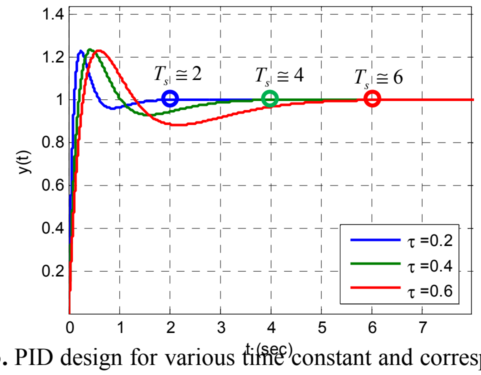

Fig. 3. PID design for various time constant and corresponding settling times.

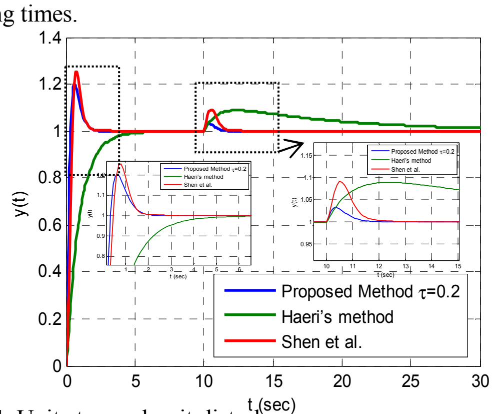

Fig. 4. Unit step and unit disturbance responses comparison of the proposed PID design method with other PID design methods for second order system model.

By using the coefficients of second-order approximate models a_0, b_0, b_1 and b_2 in the equation (24), one obtains PID design rule for SBL method as follows,

$$k_p = \frac{3(k_p(\omega_2) - k_p(\omega_1))}{\tau^2(\omega_2^2 - \omega_1^2)} - \frac{k_p(\omega_2) - k_p(\omega_1)}{\omega_2^2 - \omega_1^2}\omega_1^2 + k_p(\omega_1) \quad (25)$$

$$k_i = \frac{(k_p(\omega_2) - k_p(\omega_1))}{\tau^3(\omega_2{}^2 - \omega_1{}^2)} \quad (26)$$

$$k_d = \frac{3(k_p(\omega_2) - k_p(\omega_1))}{\tau(\omega_2{}^2 - \omega_1{}^2)} - (k_i(\omega_1)/\omega_1{}^2) \quad (27)$$

The parameter $\tau = 1/T_s$ is set to specify a desired settling time of step response. ω_1 and ω_2 are lower and upper operating frequency bounds of control system. The control system works in low frequency region. Considering equations (25)-(27), one should select $\omega_2 > \omega_1 > 0$ to obtain positive finite real controller coefficients.

3. Numerical Examples

Example 1: In this example, we design PID controller with the time constant $\tau = 0.1$ sec for the high order plant function given in equation (20) by using second-order reduced model obtained by SBL fitting.

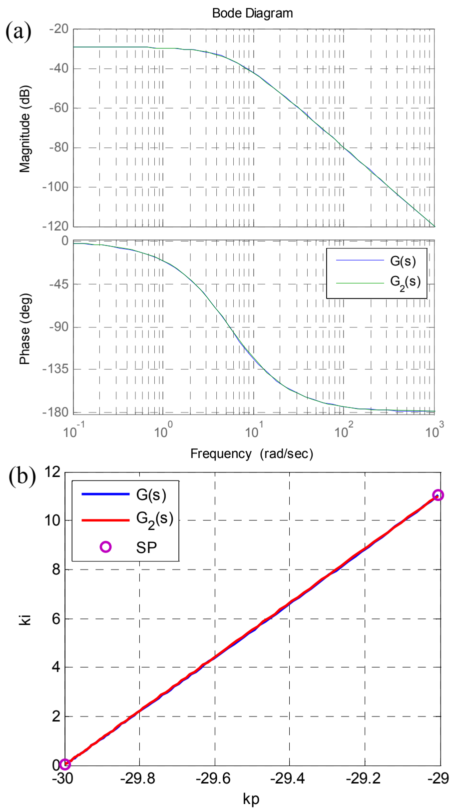

Fig. 5. (a) Bode plots of the high-order system ($G(s)$) and the second-order approximate model ($G_2(s)$); (b) SBL lines of $G(s)$ and $G_2(s)$.

Bode diagram of $G(s)$ shown in Figure 5(a) indicates that phase response of $G(s)$ is limited to -180° and the phase limitation condition of second order model reduction is satisfied. We used two sampling points for the angular frequency of $\omega_1 = 0.04$ rad/sec and $\omega_2 = 1$ rad/sec. The second-order approximate model was found as,

$$G_2(s) = \frac{1}{0.9941 s^2 + 11 s + 30} \quad (28)$$

PID design was obtained as $k_p = 268.23$, $k_i = 994.1$ and $k_d = 18.8$ for the time constant $\tau = 0.1$ sec by equations (25)-(27). Figure 5(a) reveals a satisfactory overlapping of Bode phase and magnitude plots. This demonstrates that a good frequency response matching was obtained for the both system in this example. Figure 5(b) shows SBL lines of the high-order system ($G(s)$) and the second-order approximate model ($G_2(s)$). Almost exact overlapping of SBL lines was obtained due to fact that SBL of $G(s)$ system presents a SBL curvature that suitable to implement by the second-order system dynamics.

The designed PID control system exhibits almost the same step response for the both original and approximate systems as shown in Figure 6. This indicates that model order reduction is very effective and PID control system design for the high-order plant $G(s)$ is very successful.

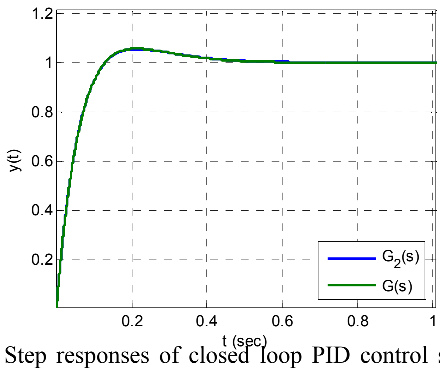

Fig. 6. Step responses of closed loop PID control system for $G(s)$ and $G_2(s)$.

Example 2: This example shows second-order SBL model identification of DC motor by stimulating closed loop PI control system for oscillation. Then, we design PID control system for the angular velocity of DC motor shaft (w_s) with settling in 0.5 sec.

DC motor control is getting significance for the rotor speed control of unmanned electrical aerial vehicle (UAV) [13] and simple DC motors are well characterized modeled by second order transfer functions. In this test, we used simple DC motor model in Simulink/MATLAB [14].

(i) The simple DC motor model in Simulink/MATLAB was connected to a closed loop PI control system.

(ii) By adjusting PI coefficients, two PI configurations oscillating the control system was found out as illustrated in Figure 7(a). Two SBL points (k_p , k_i)=(-4.098, 0.01) at $\omega_1 = 0.0847$ and (k_p, k_i) =(-4.092,0.1) at $\omega_2 = 0.2671$ cause oscillation of the closed loop control system.

(iii) The second-order model of the DC motor model was obtained by using proposed SBL fitting as,

$$G_2(s) = \frac{1}{0.0935\, s^2 + 1.394\, s + 4.099} \quad (29)$$

In order to obtain a settling time roughly 0.5 sec, the time constant should be taken as $\tau \approx 0.5/5 = 0.1$ sec. The PID design for $G_2(s)$ was obtained for this model as $k_p = 23.96$,

$k_i = 93.50$, $k_d = 1.41$. Figure 7(b) compares the results of the PID controller for the both original and identified models.

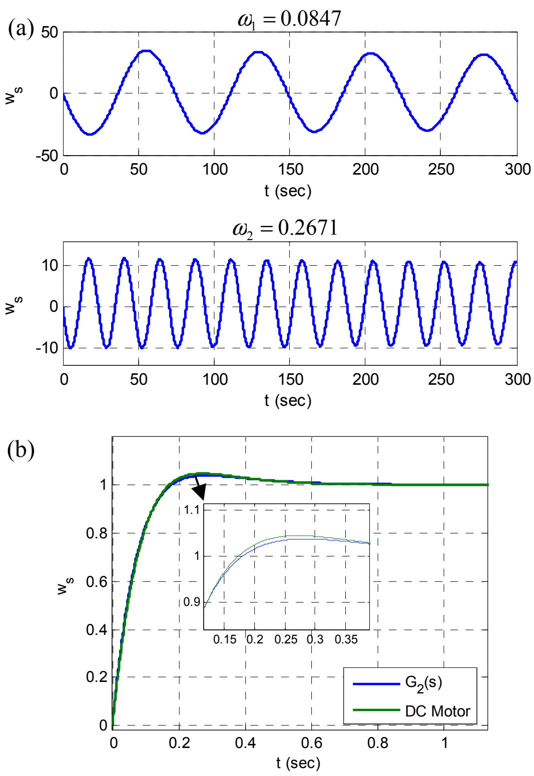

Fig. 7. (a) Oscillation of the closed loop PI control system with simple DC motor model, (b) Step responses of closed loop PID control systems for $G_2(s)$ and DC motor model.

4. Conclusions

The paper introduced a SBL based model approximation approach for the application of model order reduction and demonstrated a solution for analytical PID control system design examples for high-order systems by using model reduction. We observed that SBL model approximation method can be helpful for dealing with stability preservation problem in a desired frequency region. This property is important for controller design methods employing the model reduction.

One of the main drawbacks of the proposed analytical PID design method is that it allows PID designs for the plant complying with the phase limitation condition for second order model reduction.

Since SBL points yield oscillating system behavior, SBL approximation can be used for identification of the real plants by finding low frequency oscillation points of closed loop PI control systems. This modeling strategy can be useful for online auto-tuning of PID controllers for second order systems by detecting a couple of oscillating PI controller points.

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