Observer Based Sliding Mode Control of Uncertain Chaotic Systems

Günyaz Ablay[1] , Tunc Aldemir[2]

Nuclear Engineering Program, The Ohio State University, Columbus, Ohio, USA
[1]ablay.1@osu.edu
[2]aldemir.1@osu.edu

Abstract

A sliding mode control scheme combined with a nonlinear observer is proposed for control of uncertain chaotic systems. The control design is based on Lyapunov direct method for dealing with uncertainties with known bounds. Second and third order chaotic systems whose states are estimated through a nonlinear observer are stabilized by the sliding mode control. The numerical results are presented to verify the validity of the proposed technique.

1. Introduction

Chaos is a periodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions [1]. The fundamental characteristics of chaotic behavior come from the internal structure of the systems, and chaotic behaviors are more complicated than limit cycle behaviors. Today, chaos has been seen to have many useful applications in many engineering systems such as in chemical reactors, genetic control systems, power converters, lasers, biological systems, and secure communication systems [1]. Chaos can be useful in propagation of mixing in processes, such as in convective heat transfer. However, chaotic behavior may lead to undesirable effects as well (e.g. uncontrolled oscillations in a power grid) and may need to be regulated [2].

After chaos control was introduced in [3], it has turned out to be an important area of nonlinear science, and various control approaches have been proposed. The Ott-Grebogi-Yorke (OGY) method [3], variable structure control [4], nonlinear feedback control [5], and some other methods [6] have been successfully applied to chaotic systems. Sliding mode control (SMC) scheme is one of these methods [7-9], and recently there has been a great deal of attention on using SMC for controlling chaos. The SMC is an effective methodology for controlling systems with variable structures and provides a systematic approach to the problem of maintaining stability and consistent performance in the face of modeling imprecision and uncertainty [10-11].

Chaotic systems include nonlinearities and often some parameters which cannot be exactly defined [12]. Although some observer based control approaches have been proposed for controlling chaos [13-15], a robust control method such as SMC would have the advantage of the capability to handle such uncertainties. In the SMC, often the state trajectory of a dynamic system starting from a given set of initial conditions is desired to be controlled in such a way that it reaches a time-varying surface in a finite time, and then slides along the surface towards desired state exponentially. However, in practice, implementation of the related control switching is imperfect due to the impossibility of instantaneous switching and leads to chattering. Chattering involves high control activity and might excite high-frequency dynamics ignored during modeling. Hence, it is undesirable in practice. A common modification dealing with this problem is the boundary layer approach [10].

This work investigates an observer based SMC configuration which uses for the boundary layer approach for controlling chaos in uncertain chaotic systems, under the assumption that the uncertainties can be described through a continuous function in time. Observer based SMC is designed to stabilize second order Duffing's oscillator [16] and third order Genesio-Tesi chaotic system [17]. Section 2 overviews the use of the SMC with the boundary layer approach. Section 3 overviews observer construction procedures. In Section 4, numerical results are presented to verify the proposed method. Finally, conclusions are given in Section 5.

2. Sliding Mode Control Scheme

Consider the following single input dynamic system

$$\mathbf{x}^{(n)} = f(\mathbf{x}) + b(\mathbf{x})u \tag{1}$$

where $\mathbf{x} = [\mathbf{x} \ \dot{\mathbf{x}} \ \cdots \ \mathbf{x}^{(n-1)}]^T$ is the state vector, and $u \in R^1$ is the control input. The function $f(\mathbf{x}) \in R^n \to R^n$ is a continuous nonlinear function, but not exactly known. The control gain $b(\mathbf{x})$ is not exactly known, but is bounded by known, continuous functions of $\mathbf{x}$.

The control problem is to get the state $\mathbf{x}$ to track a specific time varying state $\mathbf{x}_d = [\mathbf{x}_d \ \dot{\mathbf{x}}_d \ \cdots \ \mathbf{x}_d^{(n-1)}]^T$ in the presence of model imprecision on $f(\mathbf{x})$ and $b(\mathbf{x})$. Therefore, the tracking error is defined as

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_d = [\tilde{\mathbf{x}} \ \dot{\tilde{\mathbf{x}}} \ \cdots \ \tilde{\mathbf{x}}^{(n-1)}]^T \tag{2}$$

where $\tilde{\mathbf{x}}$ is the tracking error vector. A time varying surface $s(\mathbf{x},t)$ in the state space R^n is defined by the scalar equation $s(\mathbf{x},t) = 0$ as

$$s(\mathbf{x},t) = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{\mathbf{x}} \tag{3}$$

where λ is a positive constant. The sliding surface is a line in the phase plane, of slope $-\lambda$ and containing the point $\mathbf{x}_d$. Figure 1 illustrates this situation graphically.

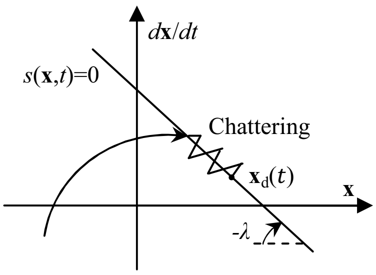

Fig. 1: Sliding surface and chattering.

The following example illustrates SMC under uncertainty in system dynamics. Consider a second order dynamic system as an example system

$$\ddot{\mathbf{x}} = f(\mathbf{x}) + u \tag{4}$$

where $f(\mathbf{x})$ is not exactly known, but estimated as $\hat{f}(\mathbf{x})$ and u is the control vector. The estimation error on $f(x)$ is assumed to be bounded by some known function $F = F(\mathbf{x}, \dot{\mathbf{x}})$

$$\left|\hat{f}(\mathbf{x}) - f(\mathbf{x})\right| \le F \tag{5}$$

To have the system track $\mathbf{x}(t) \equiv \mathbf{x}_d(t)$, a sliding surface is defined by $s \equiv s(\mathbf{x}, t) = 0$, that is,

$$s = \dot{\tilde{\mathbf{x}}} + \lambda\tilde{\mathbf{x}} = 0 \tag{6}$$

$$\Rightarrow \dot{s} = \ddot{\tilde{\mathbf{x}}} + \lambda\dot{\tilde{\mathbf{x}}} = \ddot{\mathbf{x}} - \ddot{\mathbf{x}}_d + \lambda\dot{\tilde{\mathbf{x}}} = f(\mathbf{x}) + u - \ddot{\mathbf{x}}_d + \lambda\dot{\tilde{\mathbf{x}}} \tag{7}$$

The best approximation $\hat{u}$ of a continuous control law that would achieve $\dot{s} = 0$ is thus

$$\hat{u} = -\hat{f}(\mathbf{x}) + \ddot{\mathbf{x}}_d - \lambda\dot{\tilde{\mathbf{x}}} \tag{8}$$

where $\hat{u}$ is well-known as the equivalent control. To satisfy sliding condition in the presence of uncertainty on the dynamics, a discontinuous term is added to the equivalent control.

$$u = \hat{u} - k\,sign(s) \tag{9}$$

where $sign(s)$ is the sign function defined as $sign(s) = 1$ if $s > 0$, $sign(s) = -1$ if $s < 0$.

For stability analysis, a Lyapunov function is defined as

$$V(s) = 0.5s^2 \tag{10}$$

The derivative of (10) must be negative for stability. Then

$$\frac{1}{2}\frac{d}{dt}s^2 = s\dot{s} = [f(\mathbf{x}) - \hat{f}(\mathbf{x}) - k\,sign(s)]s = [f(\mathbf{x}) - \hat{f}(\mathbf{x})]s - k|s|$$

and letting $k = F + \eta$ where $\eta > 0$, the following equation is obtained which indicates stability [11]:

$$\frac{1}{2}\frac{d}{dt}s^2 \le -\eta|s| \tag{11}$$

Controller design procedure consists of two steps. In the first step, a state feedback control u is selected to ensure sliding condition or stability. The second step is related to suitably smoothing the discontinuous control to eliminate chattering problem, and to achieve robustness against high frequency unmodeled dynamics. For this reason, the saturation function, $sat(s/\beta)$ where β is the thickness of the boundary layer, is usually used instead of $sign(s)$.

$$sat(s/\beta) = \begin{cases} -1 & if \quad s \le -\beta \\ s/\beta & if \quad -\beta < s \le \beta \\ 1 & if \quad \beta < s \end{cases}$$

Therefore, the control law has the following form

$$u = \hat{u} - k\,sat(s/\beta)\,. \tag{12}$$

3. Nonlinear Observer

Consider the following single input, single output nonlinear system.

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + Bu + \varphi(\mathbf{x}) \\ y = C\mathbf{x} \end{cases} \tag{13}$$

where $\mathbf{x} \in R^n$ is the state variable, $u \in R$ is the control vector, $A \in R^{nxn}$ is the system matrix, $B \in R^{nx1}$ is the control matrix, $\varphi(\mathbf{x}): R^n \to R^n$ is a nonlinear function, $y \in R$ is the output state, and $C \in R^{1xn}$ is the output matrix.

A necessary and sufficient condition for observability in the system given in Eq.(13) is that the rank of the matrix

$$N = [\,C^T \quad A^T C^T \quad \cdots \quad (A^T)^{n-1} C^T\,] \tag{14}$$

is n, namely $rank(N) = n$. The most common observer scheme involves replicating the system dynamics together with an additive output error injection term. A Luenberger-like observer for nonlinear systems, the Thau observer [18], can be defined as

$$\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + Bu + \varphi(\hat{\mathbf{x}}) + L(y - C\hat{\mathbf{x}}) \tag{15}$$

where $\hat{\mathbf{x}} \in R^n$ is the observer state vectors, $L \in R^{nx1}$ is the observer gain matrix, and $C\hat{\mathbf{x}} \in R$ is the observer output.

The observer gain matrix L can be computed by using pole placement method. For an appropriate choice of L, the eigenvalues of (A-LC) can be negative, and the observer error, $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$, can converge to zero asymptotically; that is, $\lim_{t\to\infty}\|\mathbf{e}\| = 0$. As a result of this, the observer state vectors may be replaced the state variables of the system. For details see [18] and [19].

4. Control of the Chaotic Systems Using Observer Based SMC

Control of the chaotic systems using observer based SMC is illustrated with the Duffing's Oscillator and the Genesio-Tesi system.

4.1. The Duffing's Oscillator

The fundamental Duffing's Oscillator is defined by

$$\ddot{x} + c\dot{x} + dx^3 = \theta\cos(wt) + u \tag{16}$$

Assume the forcing function $q(t) = \theta\cos(wt)$ has uncertainty in θ with $\theta_1 \le \theta \le \theta_2$. Introducing the state vectors $x_1 = x$, $x_2 = \dot{x}$, and rewriting Eq.(16) in the state space form yields the controlled Duffing's oscillator

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -dx_1^3 - cx_2 + q(t) + u \end{cases} \tag{17}$$

where u is the control input. The system output is defined by

$$y = [1\ \ 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{18}$$

where y is the output state. Generally, it is desired that error states converge to zero, that is $\mathbf{x}_d = 0$. Hence, by using observer state vectors given in (15), the tracking error given in (2) has the following form.

$$\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}_d = [\hat{x}_1\ \hat{x}_2]^T . \tag{19}$$

Sliding surface is defined by

$$s = \hat{x}_2 + \lambda\hat{x}_1 \tag{20}$$

and

$$\dot{s} = \lambda\hat{x}_2 - d\hat{x}_1^3 - c\hat{x}_2 + q(t) + u .$$

The equivalent control $\hat{u}$ which is the solution of $\dot{s} = 0$ is found as

$$\hat{u} = -(-d\hat{x}_1^3 - c\hat{x}_2 + \hat{q}(t)) - \lambda\hat{x}_2$$

where $\hat{q}(t)$ is the estimated value of the forcing term $q(t)$. The control law can be selected as given in (12):

$$u = \hat{u} - ksat(s/\beta) \tag{21}$$

The parameters in (16) are chosen as $c = 0.05$, $d = 1$ and $2.5 \le \theta \le 9.5$, and the forcing function is selected as $q(t) = \theta\cos(2\pi t)$ in order to obtain a chaotic behavior. The initial conditions are chosen as $\mathbf{x}(0) = [3,\ 4]^T$. The other parameters and functions are selected as $\lambda = 1$, $\eta = 9$, $\beta = 0.05$ and $\hat{q}(t) = 9.5\cos(2\pi t)$.

To design the observer, when desired observer eigenvalues are selected as $p = [-5\ \ -5]^T$, by using pole placement method, the observer gain matrix can be obtained as

$$L = [9.9510\quad 24.5074]^T$$

The initial conditions of the observer are taken as $\hat{\mathbf{x}}(0) = [1,\ -2]^T$, and the uncertainty in the parameter θ is described through $\theta = 7|\sin t| + 2.5$.

The simulation results are provided in Figures 2 - 5 to verify the proposed method. The time responses of the state variables of the Duffing's oscillator are given in Figure 2. The time responses of the observer error vectors $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$ are displayed in Figure 3. It is seen that the observer state vectors converge to the state vectors of the Duffing's oscillator quickly.

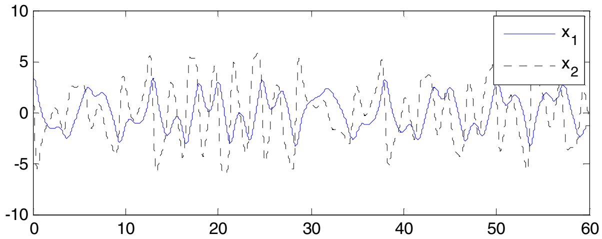

Fig. 2: The time responses of the state variables of the Duffing's oscillator for u=0.

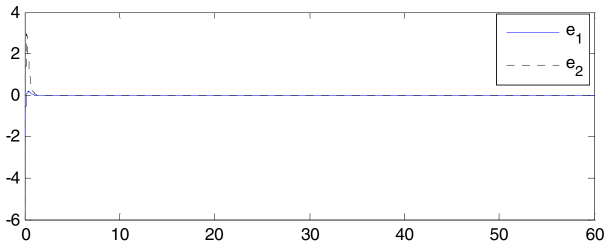

Fig. 3: The time responses of the observer error vectors for u=0.

In Figures 4 – 5, the controller and observer are activated at the time $t = 30$. In Figure 4, the time responses of the state variables of the Duffing's oscillator are given. It is seen that the state variables of the system converge to zero immediately after the controller is activated. Figure 5 depicts the control signal. It is obvious that the chattering in the control signal is eliminated by using boundary layer approach.

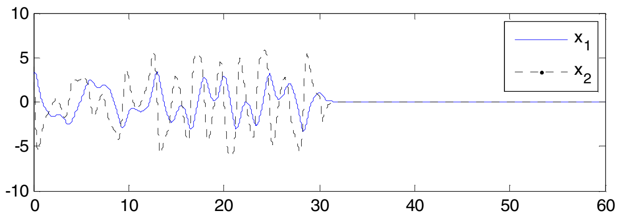

Fig. 4: The time responses of the state variables of the Duffing's oscillator with control activated at $t = 30$.

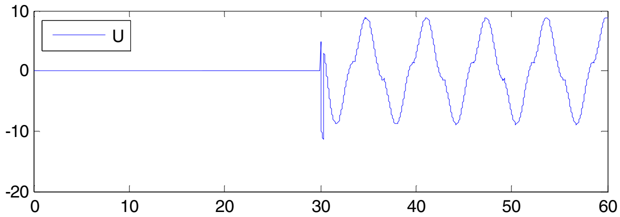

Fig. 5: The time response of the control signal *u*.

4.2. The Genesio-Tesi System

Genesio-Tesi system with a control input is defined through the following equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -cx_1 - bx_2 - ax_3 + dx_1^2 + u \end{cases} \tag{22}$$

where $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^T$ is the state variables, u is the control input, and the parameters *a*, *b*, *c*, *d* are not exactly known, but are bounded. The system output is defined as

$$y = [1 \;\; 0 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{23}$$

where *y* is the output state. For $\mathbf{x}_d = 0$, by using observer state vectors given in Eq.(15), the tracking error is

$$\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}_d = [\hat{x}_1 \;\; \hat{x}_2 \;\; \hat{x}_3]^T \tag{24}$$

and the sliding surface from Eq.(3) is

$$s = \hat{x}_3 + 2\lambda\hat{x}_2 + \lambda^2\hat{x}_1 \tag{25}$$

$$\Rightarrow \dot{s} = -c\hat{x}_1 - b\hat{x}_2 - a\hat{x}_3 + d\hat{x}_1^2 + u + 2\lambda\hat{x}_3 + \lambda^2\hat{x}_2$$

The equivalent control $\hat{u}$ is found as

$$\hat{u} = -\hat{f}(\hat{x}) - 2\lambda\hat{x}_3 - \lambda^2\hat{x}_2$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are the estimated values of the parameters a, b, c, and d. The control law is

$$u = \hat{u} - k sat(s/\beta) \tag{26}$$

Suppose that the parameters of the system can vary as $0.2 \le a \le 0.4$, $1 \le b \le 1.8$, and $0.5 \le c = d \le 1.1$ (while still maintaining chaotic behavior). The initial conditions and the other values are selected as follows: $\mathbf{x}(0) = [0.5,\ 0,\ 0]^T$, $\lambda = 1$, $\eta = 0.1$, $\beta = 0.1$, $\hat{f}(\hat{x}) = -\hat{x}_1 - 1.5\hat{x}_2 - 0.3\hat{x}_3 + 0.9\hat{x}_1^2$ and $F = 0.5\hat{x}_1 + 0.5\hat{x}_2 + 0.1\hat{x}_3 + 0.4\hat{x}_1^2$.

By assuming the nominal parameter values $a = 0.25$, $b = 1.4$, $c = d = 1$, and choosing the desired observer eigenvalues as $p = [-3 \;\; -3 \;\; -3]^T$, the observer gain matrix by using pole placement method can be obtained as

$$L = [8.75 \quad 23.4125 \quad 7.8969]^T$$

In numerical simulations, the observer initial conditions are taken as $\hat{\mathbf{x}}(0) = [0.1,\ 0,\ 0]^T$, and the parameters *a*, *b*, *c*, *d* are defined as a function of sine, that is, $a = 0.2|\sin t| + 0.2$, $b = 0.8|\sin t| + 1$, and $c = d = 0.6|\sin t| + 0.5$. The simulation results are presented in Figures 6 – 9.

In Figure 6, the time responses of the state variables of the Genesio-Tesi system is given with the control signal *u*=0. The time responses of the observer error vectors $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$ are illustrated in Figure 7.

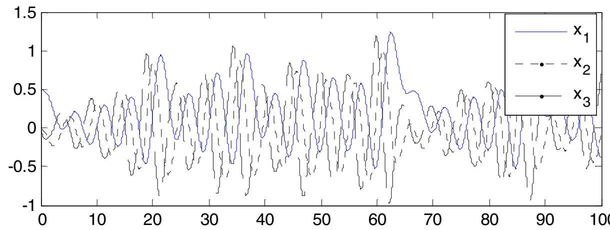

Fig. 6: The time responses of the state variables of the Genesio-Tesi system with *u*=0.

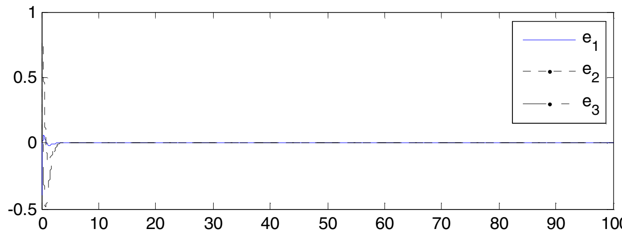

Fig. 7: The time responses of the observer error vectors **e** with *u*=0.

In Figures 8 – 9, the controller and observer are activated at the time $t = 40$. In Figure 8, the time responses of the state variables of the Genesio-Tesi system is given. In Figure 9, the control signal is given. It is clear that the chattering in the control signal is eliminated via boundary layer approach.

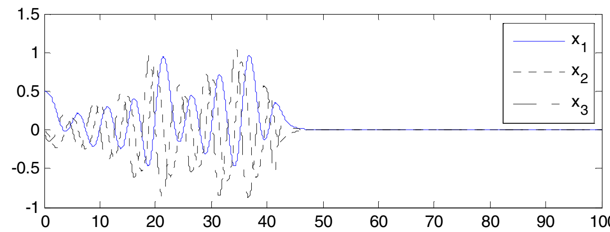

Fig. 8: The time responses of the state variables of the Genesio-Tesi system with control activated at $t = 40$.

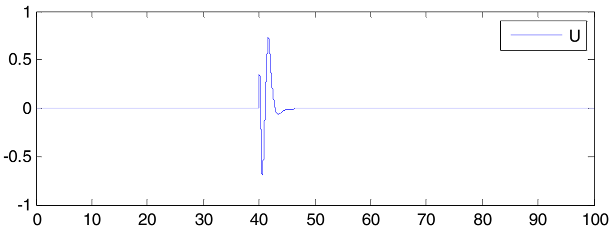

Fig. 9: The time response of the control signal u.

5. Conclusions

This study investigates observer based control of uncertain chaotic systems by using the sliding mode control method. A nonlinear observer and the SMC scheme are designed in the presence of the unknown but bounded parameter values. The uncertainties are assumed to be describable through a continuous function in time. The chattering problem of the SMC is eliminated with the boundary layer approach.

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7. References

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