INDUCTANCE OF COAXIAL CABLE

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Key words: Electromagnetism, skin effect, numerical analysis

ABSTRACT

The cable is formed by two tubular conductors and is connected to a sinusoidal voltage source whose frequency is less than 10 MHz. A method for calculating the inductance of this cable is proposed. The method is applied on examples and the results obtained are compared with published data.

I. INTRODUCTION

Consider a coaxial cable formed by two infinitely long straight concentric tubular conductors. The inner or the outer conductor is determined by resistivity and cross-section radii ϱ_i, r_{i1} and r_{i2} or ϱ_o, r_{o1} and r_{o2}. The permeability of conductors and the insulation between them is assumed to be equal to μ_0. In addition to the Cartesian system the system of cylindrical coordinates r, ϕ, z is also used. But because of the symmetry with respect to axis z the ϕ coordinate will be to no avail. The two conductors have the direction of axis z and are connected to an ideal voltage source [1]. An electrical appliance is connected to their other end. The source voltage and current I in the conductors depend only on time $t \in [0, \infty)$. The ratio of conductor cross-section magnitudes is denoted

$$q = \frac{r_{o2}^2 - r_{o1}^2}{r_{i2}^2 - r_{i1}^2}. \tag{1}$$

The complicated motion of electric charges forming a current in the conductor is described by the vector of current density $\boldsymbol{J}$. In view of the symmetry of the arrangement of conductors it is assumed that $\boldsymbol{J}$ has the direction of axis z and depends only on r and t;

$$\boldsymbol{J} = \begin{array}{ll} (0, 0, J(r,t)) & \text{for} \quad r \in [r_{i1}, r_{i2}], \\ (0, 0, -J(r,t)) & \text{for} \quad r \in [r_{o1}, r_{o2}], \end{array}$$

where function $J = J(r,t)$ is defined for $r \in [0, r_{o2}]$ and

$$J(r,t) = 0 \quad \text{for} \quad r \in [0, r_{i1}) \cup (r_{i2}, r_{o1}).$$

The magnitude B of field vector $\boldsymbol{B}$ also depend on only r and t. The fact that current I and vectors $\boldsymbol{J}, \boldsymbol{B}$ do not depend on z actually means that an infinitely high propagation velocity of electromagnetic field is assumed. The section of each of the conductors through the plane $z = \text{const}$ is an equipotential area. In the whole of this paper the part of the two conductors between the planes $z = z_1$ and $z = z_2$, where $z_1 < z_2$, $z_{21} = z_2 - z_1 = 1$ m, will be considered. Across the cross-sections of the two conductors in the plane $z = z_1$ or $z = z_2$ is a voltage $U_1 = U(z_1, t)$ or $U_2 = U(z_2, t)$.

II. CURRENT DENSITY IN CONDUCTORS

If the cable is connected to a source whose voltage varies with time, then the current density in the cable cross-section is affected by induced currents. In this Section an equation is derived in brief the solution of which is the current density in the cable for a given time dependence of the voltage source. A detailed derivation and the solution method are given in [2].

Coaxial conductors are formed by elementary conductors, which form elementary loops. The cross-section of elementary conductors is $\mathrm{d}A_c = 2\pi\, r_c\, \mathrm{d}r_c$, where subscript c denotes either i or o. While r_i is an arbitrary number from the interval $[r_{i1}, r_{i2}]$, r_o must be chosen such that the same current $\mathrm{d}I$ flows through cross-sections $\mathrm{d}A_i$ and $\mathrm{d}A_o$, and the mapping, which to each $r_i \in [r_{i1}, r_{i2}]$ assigns $r_o \in [r_{o1}, r_{o2}]$, must be a one-to-one mapping of the intervals $[r_{i1}, r_{i2}]$ and $[r_{o1}, r_{o2}]$. A mapping fulfilling these conditions will be denoted by the letter m, i.e. $r_o = m(r_i)$. To determine m means to determine the lines of vector $\boldsymbol{J}$ in the conductors. By analogy with relation (1) it is assumed that it holds

$$\mathrm{d}A_o = q\, \mathrm{d}A_i. \tag{2}$$

The current flowing through an elementary loop is $\mathrm{d}I$ so that

$$J(r_i,t)\,\mathrm{d}A_i = J(r_o,t)\,\mathrm{d}A_o.$$

From this relation and from (2) it follows that

$$J(r_o,t) = \frac{J(r_i,t)}{q}. \tag{3}$$

Each elementary loop will be replaced by a lumped-elements circuit for which it holds by Kirchhoff's voltage law

$$U_i(r,t) + U_o(r,t) + U_L(r,t) = U_{12}. \tag{4}$$

The voltage drop $U_{12} = U_1 - U_2$ is the same along each segment (1 m long) of both conductors. We will in the following assume that

$$U_{12} = \widehat{U} u(t), \tag{5}$$

where the function $u(t)$ is dimensionless while $\widehat{U}$ is a constant expressed in volts. U_i and U_o are voltages across the resistances that substitute the elementary conductors. Using (3) we obtain

$$U_i + U_o = \left(\varrho_i + \frac{\varrho_o}{q}\right) z_{21}\, J(r_i,t). \tag{6}$$

The voltage U_L in equatiom (4) is a voltage induced around the elementary loop. Faraday's law of induction is expressed by Maxwell's equation in integral form

$$\oint_C \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{C} = -\frac{\mathrm{d}\varPhi}{\mathrm{d}t}.$$

The integral of electric field intensity vector around a closed curve C is the electromotive force (emf) around C while $\varPhi$ is the flux of the magnetic field vector $\boldsymbol{B}$ through a continuous surface A_C bounded by the curve C. $\varPhi$ does not depend on the shape of the surface A_C [3]. For the calculation of U_L the surface A_C may be a rectangle $[r_i,r_o]\times[z_1,z_2]$ in a plane $y=0$, and the curve C is then the boundary of this rectangle. Taking into consideration the fact that the magnetic field $\boldsymbol{B}$, excited by the current in the pair of conductors under consideration, has on the circle $r = \mathrm{const}$ the direction of tangent to this circle, it follows from the above that

$$U_L = \frac{\mathrm{d}\varPhi}{\mathrm{d}t}, \tag{7}$$

where

$$\varPhi = \int_{A_C} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{A}_C = z_{21}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_{r_i}^{r_o} B(x, J(x,t))\,\mathrm{d}x.$$

Using x instead of r only says that in the calculation of $\varPhi$ it is only the values of B on axis x that come to be applied. B is easy to calculate using Ampere's circuital law.

Using (5), (6) and (7), equation (4) can be rewritten in the form

$$\left(\varrho_i + \frac{\varrho_o}{q}\right) J(r_i,t) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{r_i}^{r_o} B(x, J(x,t))\,\mathrm{d}x = \frac{\widehat{U}u(t)}{z_{21}}.$$

This equation must be satisfied for any $r_i \in [r_{i1}, r_{i2}]$ and $t \in [0,\infty)$. Its solution determines the current density $J(r,t)$ in the above pair of coaxial conductors for a given $\widehat{U}u(t)$. The solution $J(r,t)$ is directly proportional to $\widehat{U}$.

III. INDUCTANCE OF COAXIAL CABLE

The inductance of 1 metre of cable can be determined on the assumption that the cable is connected to a source of sinusoidal voltage, i.e. $u(t) = \sin\omega t$, is assumed. In that case it is of advantage to solve examples for the complex voltage

$$\underline{U}_{12}\exp(\mathrm{j}\omega t)$$

and for the current density

$$\underline{J}(r)\exp(\mathrm{j}\omega t). \tag{8}$$

The underlined symbols denote phasors

$$\begin{aligned}\underline{U}_{12} &= \widehat{U},\\ \underline{J}(r) &= \widehat{J}(r)\exp(\mathrm{j}\alpha(r))\end{aligned}$$

and thus

$$J(r,t) = \widehat{J}(r)\sin[\omega t + \alpha(r)]. \tag{9}$$

The equivalent circuit of 1 metre of cable is the series connection of a resistor of resistance

$$R = \frac{z_{21}}{\pi(r_{i2}^2 - r_{i1}^2)}\left(\varrho_i + \frac{\varrho_o}{q}\right),$$

which is the sum of the resistance of 1 metre of inner conductor and the resistance of 1 metre of outer conductor, and an inductor characterized by inductance L. In addition to the magnetic field excited by the current in the cable the effect of other magnetic fields is not taken into consideration. The displacement current in the insulation between the conductors is also neglected, which leads, as shown in [2], to the source frequency being limited by the condition $f < 10$ MHz.

The current density (8) calculated for a given voltage $\widehat{U}\exp(\mathrm{j}\omega t)$ uniquely determines the current through the cable

$$\underline{I}\exp(\mathrm{j}\omega t).$$

By Kirchhoff's voltage law it holds

$$\underline{I}\,(R + \mathrm{j}\,\omega L) = \widehat{U}. \tag{10}$$

As mentioned above, the current density in the cable is directly proportional to the source voltage and therefore the current through the cable is also directly proportional to the source voltage and we can assume that $\widehat{U} = 1$ V. Therefore it follows from (10) that

$$L = -\frac{\sin[\arg(\underline{I})]}{\omega\,|\underline{I}|}. \tag{11}$$

IV. EXAMPLE

The inner copper conductor ($\varrho_i = 1.712 \times 10^{-8}$ Ω·m) has the dimensions $r_{i1} = 5$ mm, $r_{i2} = 10$ mm; the outer conductor is of aluminium ($\varrho = 2.709 \times 10^{-8}$ Ω·m) with $r_{o1} = 11$ mm, the ratio of conductor cross-section magnitudes $q = 1.5$; $\omega = 2\pi f$, where $f = 60$ Hz and 1 kHz. Values of ϱ_c are taken from [4]. Figure 1 shows the normalized amplitude of the current density

$$\frac{\widehat{J}(r)}{J_{\max}}, \quad \text{where} \quad J_{\max} = \widehat{J}(r_{i2}),$$

in the inner and the outer conductors, $J_{\max} = 2.82032 \times 10^7$ A/m^2 for $f = 60$ Hz and $J_{\max} = 2.28936 \times 10^7$ A/m^2 for $f = 1$ kHz. Figure 2 shows the value $\alpha(r)$ (see (9)). The current (in amperes) in the inner and the outer conductor is

$$I(t) = \begin{array}{ll} 6546.86 \sin(\omega t - 10.89\,\mathrm{deg}) & \text{for } f = 60 \text{ Hz} \\ 2098.01 \sin(\omega t - 55.56\,\mathrm{deg}) & \text{for } f = 1 \text{ kHz} \end{array}$$

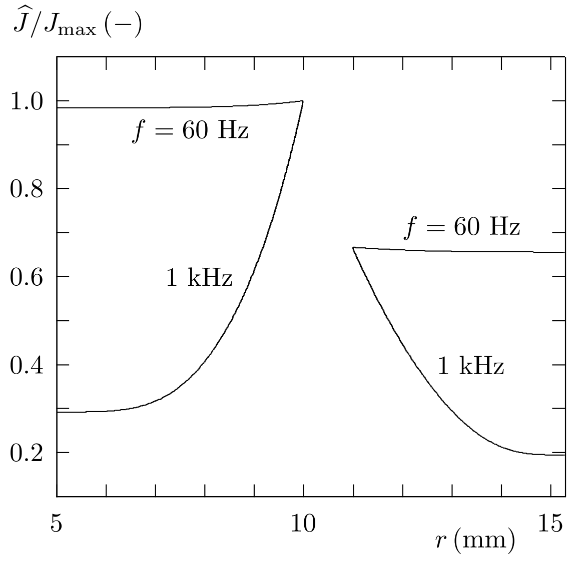

Figure 1. Normalized amplitude of the current density in conductors.

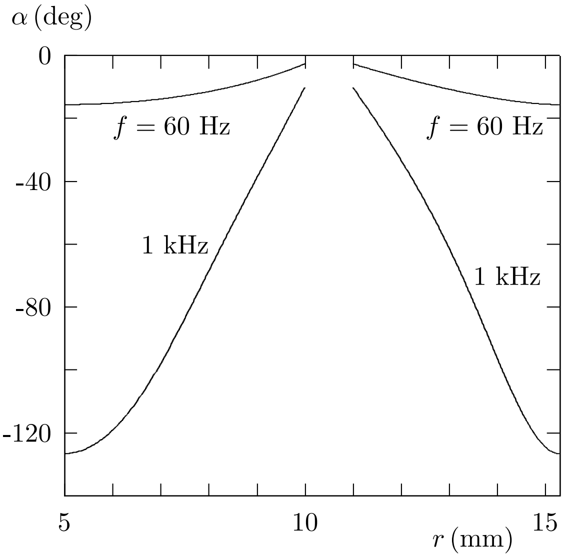

Figure 2. Initial phase angle $\alpha(r)$ of the current density in conductors.

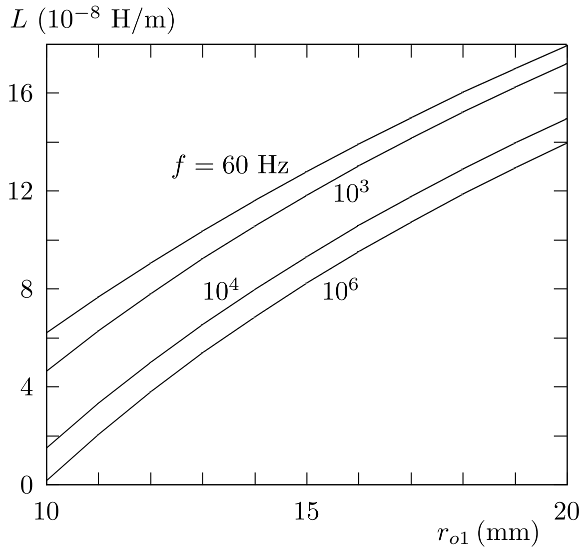

Figure 3. Dependence of inductance L on f and r_{o1}, calculated using formula (11).

V. COMPARISON OF RESULTS

The results of calculating L using the relation (11), in which phasor $\underline{I}$ is determined by a method described in Section II, will be compared with the values L_{H} and L_{S}. The formula for calculating L_{H} is given in [5] for $r_{i1} = 0$. For the high frequencies $f = 1$ to 100 MHz

$$L_{\mathrm{H}} = \frac{\mu_0}{2\pi} \ln \frac{r_{o2}}{r_{i2}} \tag{12}$$

for the low frequencies it is necessary to add $5{\times}10^{-8}$ H/m on the right-hand side of the relation

(12). In [6] the formula for the calculation of inductance is derived on the assumption that $r_{i1} = 0$, $r_{o2} = \infty$

$$L_\mathrm{S} = \frac{\mu_0}{2\pi}\left[\ln\frac{r_{o1}}{r_{i2}} + \sqrt{\frac{\varrho_i}{2\omega\,\mu_0}}\left(\frac{1}{r_{i2}} + \frac{1}{r_{o1}}\right)\right]. \qquad (13)$$

Inductance will be compared for a cable of copper conductors ($\varrho = 1.712 \times 10^{-8}$ Ω·m), the size of the cross-section of outer conductor is equal to the size of the cross-section of inner conductor; $r_{i1} = 0$, $r_{i2} = 5$ mm, $r_{o1} = 5, 6, 7, 8, 9, 10$ mm, $f = 50$ Hz and 1 MHz. The values of inductance expressed in H/m and the percent deviations of L_H and L_S from L are given in Table 1.

thinner. For example, in the case of $r_{o1} = 7$ mm, $f = 1$ MHz is the maximum value of current density $J_\mathrm{max} = 1.154\times10^6$ A/m^2 for $r = r_{i2}$ and $r = r_{o1}$ while over most of the cross-section its value is with respect to J_max negligible. If we choose $r_{i1} = 4$ mm, the mass of cable conductor drops to 36 % of the initial mass, the current magnitude does not change (for $U = 1$ V), J_max slightly increases to 1.282×10^6 A/m^2, and there is a slight change in inductance L to 6.959×10^{-8} H/m while L_H and L_S remain the same as in the Table for $r_{i1} = 0$. The unsuitability of formula (13) for the low frequencies follows from the fact that

$$\lim_{\omega\to0+} L_\mathrm{S} = +\infty$$

Table 1. Comparison of inductances.

f (Hz)	r_{o1} (mm)	L (H/m)	L_H (H/m)	Δ (%)	L_S (H/m)	Δ (%)
50	5	7.723×10^{-8}	1.193×10^{-7}	54.5	3.725×10^{-7}	382.3
	6	1.064×10^{-7}	1.392×10^{-7}	30.8	3.779×10^{-7}	255.2
	7	1.325×10^{-7}	1.585×10^{-7}	19.6	3.866×10^{-7}	191.8
	8	1.559×10^{-7}	1.770×10^{-7}	13.5	3.967×10^{-7}	154.5
	9	1.771×10^{-7}	1.945×10^{-7}	9.8	4.073×10^{-7}	130.0
	10	1.965×10^{-7}	2.109×10^{-7}	7.3	4.180×10^{-7}	112.7
10^6	5	2.682×10^{-9}	6.931×10^{-8}	2484.0	2.634×10^{-9}	-1.8
	6	3.893×10^{-8}	8.920×10^{-8}	129.1	3.888×10^{-8}	-0.1
	7	6.961×10^{-8}	1.085×10^{-7}	55.9	6.955×10^{-8}	0.0
	8	9.622×10^{-8}	1.270×10^{-7}	32.0	9.614×10^{-8}	0.0
	9	1.197×10^{-7}	1.445×10^{-7}	20.7	1.196×10^{-7}	0.0
	10	1.407×10^{-7}	1.609×10^{-7}	14.4	1.406×10^{-7}	0.0

As can be seen from the Table, the formula (12) only gives orientational values of the cable inductance because when it was being derived a constant current density in the cross-section of conductors was assumed and part of the magnetic flux was neglected so that also the dependence on f is vague. The formula (12) is acceptable for the low frequencies in the case that the size of conductor cross-section is small with repect to the size of the cross-section of insulation between the conductors. For example, if for $f = 50$ Hz, $r_{o1} = 10$ mm we choose $r_{i2} = 1$ mm, then $L_\mathrm{H} = 5.115 \times 10^{-7}$ H/m, $L = 5.108 \times 10^{-7}$ H/m and for $r_{i2} = 0.3$ mm we have L_H identical to L in the first four valid figures.

A comparison of the inductance values in the Table reveals that the formula (13) is more accurate for the higher frequencies. The inductance L_S, calculated using the (13) formula, does not depend on r_{i1} and r_{o2}, which is not hindrance exactly at the higher frequencies. At the higher frequencies the distribution of current density over the conductor cross-section markedly depends on the distance of r from the conductor axis. With increasing frequency the layer which is close to the conductor surface ($r = r_{i2}$, $r = r_{o1}$) and though which most of the current flows becomes

but, for example, for $r_{o1} = 7$ mm

$$\lim_{\omega\to0+} L = 1.325 \times 10^{-7}\ \mathrm{H/m}$$

and $L_\mathrm{H} = 1.585 \times 10^{-7}$ H/m (see the Table). The reason is that inductance is defined independently of f and must thus make sense also for $f = 0$.

In the calculation of inductance using the relation (11) the method for calculating current density is of fundamental importance. One of the advantages of the latter method is that the current density can also be determined for $r_{i1} > 0$. This advantage shows in particular in the calculation of inductance at the lower frequencies, when the magnitude of current density is non-negligible over the whole cross-section of conductors. For example, if for $f = 50$ Hz and $r_{o1} = 7$ mm we change r_{i1} from zero to 4 mm, L changes from the value 1.325×10^{-7} H/m (see the Table) to 8.643×10^{-8} H/m while the values L_H and L_S remain unchanged.

Both in the proposed method and when deriving formulae for the calculation of L_H and L_S, the conductors were assumed to be solid. A number of works have been published, e.g. [7–10], where inductance is computed by the so-called wire model, in which the cross-section is subdivided into smaller segments that are individually replaced with wire elements. In all these works, however, conductor resistance is assumed to be dependent on the source frequency, as derived in [11] for solitary cylindrical conductor. In [2] the frequency dependence of resistance was called into question and so was the appropriatness of the solitary conductor conception in the computation of inductance. In view of the above, the wire model has not received much attention in the present paper.

VI. CONCLUSION

A brief description was given of a method for calculating the current density in a coaxial cable connected to the source of time-dependent voltage. The cable is formed by two tubular conductors, which can be of different resistivity. The method for calculating the current density could be used, without any major changes, also in the case that each conductor is formed by several layers of different resistivity. Using the current density and relation (11) the inductance of 1 meter of cable connected to a source of sinusoidal voltage can be calculated if the voltage frequency is below 10 MHz. The method proposed is used in the solution of two examples. In the second example the inductance determined by the proposed method is compared with inductances calculated using the (12) and (13) formulae given in the literature. This comparison yields certain applicability limits for the (12) and (13) formulae.

ACKNOWLEDGMENTS

This paper contains solution results of the Research plan No MSM0021630516 of the Ministry of Education, Youth and Sports of the Czech Republic.

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