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On The Formulation Of Equations For Mechanical Systems In Planar Motion

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Abstract

The purpose of this paper is to present the possibility of obtaining the equations of motion for the mechanical systems in the form of state equations by using the methods well established in electrical engineering. After giving a brief summary about this approach, three examples are discussed in detail. To keep the discussion rather simple, the mechanical systems used in these examples are assumed to be in planar motion. However, the same technique is also applicable if a mechanical system is in three dimensional motion.

1. Introduction

One of the well known techniques used widely in engineering to formulate the equations of motion of a given mechanical system is the Lagrangian formulation. This formulation which yields the Lagrangian equations of motion for the system relies heavily on the energy functions defined for that system which are expressed in terms of the generalized coordinates [1]. However, the analogy that exists between the mechanical systems and the electrical networks permits one to obtain an equivalent set of equations of motion for the mechanical systems by some of the techniques well established and used in electrical network theory. The analogy is perfect if the mechanical system is in one dimensional motion. In the general case, however, the distinction occurs in the nature of the terminal variables : In electrical networks they are scalars while in mechanical systems they must be treated as vectorial quantities.

The purpose of this article is to furnish a network approach to the formulation of the equations of motion of a class of mechanical systems in planar motion. Here, focusing the attentions to the planar motion of mechanical systems does not bring any major restriction to the generality of the method. Indeed, in an earlier publication [2], a general mathematical model for a rigid body in three dimensional motion, as a multiport component, is obtained by the application of this approach. The same network approach is used to derive the equations of motion of a system of rigid bodies connected in some special way to form an open kinematical chain, a configuration used in a large variety of manipulators [3]. In the present paper, after introducing a short summary of the network approach used in [2] and [3], application of this technique is extended to the systems in which mechanical components other than the rigid bodies are also contained.

However, for the sake of simplicity it is assumed that components such as two-terminal springs and dampers are in one dimensional motion. This restriction actually is met in the case of vibrating systems oscillating about positions of an equilibrium with a small magnitude. Examples to clarify the use of this procedure are given.

2. The Method

Analysis of a physical system with lumped components, whether the system contains pure electrical or pure mechanical components only, necessitates a knowledge for the characterization of each component included in the system and also a knowledge for the interconnection pattern (configuration) of these components as to how the system is formed [4],[6]. Component characterization is actually a postulated terminal representation which constitutes the mathematical model of the component. This model possesses two important features: (1) an oriented *terminal graph* which is a tree or a collection of trees, indicating the ports of the component and the orientations of the instruments, real or conceptual, connected at these ports to measure a pair of complementary variables (one across and one through) to describe the physical properties of the component, and (2) the *terminal equations* or the *constitutive equations*, yielding the relationships between all the measured across and through variables at the ports [6],[8]. However, without violating these essential features of a component, some redundant terminal representation may also be defined and conveniently be used [9].

On the other hand, interconnection pattern of the components is best represented by an oriented graph (a *system graph*) which allows one to write in a systematical manner the relationships (Kirchoff's equations) that must be satisfied by the terminal variables associated with the components forming the system. These relations being linear and algebraic in nature are referred to, in network and system theory, as the *circuit* and *cut-set* equations. For the given system, once a mathematical model for each component and the interconnection pattern of the components are known, then the system equations can be obtained systematically in variety of different forms. Amongst these forms, the *state equations* are preferable because of their suitability for computer solutions. The state variables for a mechanical system are the velocities of two-terminal mass or inertia components and also the forces or torques (moments) of two-terminal springs. Generally not all mass or inertia velocities and not all spring force and torque variables are available as the state variables. The state variables are determined by selecting a *proper tree*, T, in the system graph which should include as many edges as possible corresponding to the two-terminal mass or inertia components and , if possible, all the edges corresponding to the two-terminal springs (translational or rotational) should be included in the complementary graph (*co-tree*) , T', of T. The edges of T are called *branches* while the edges of T' are called *chords* [10].

Although a mechanical system in three-dimensional motion can be studied by the linear graph approach which gives equations of motion in the form other than the state equations [11], in establishing the state equations of a given system directly, it is found more convenient to represent multiport components, at least conceptually, as the interconnected two-terminal (one-port) components together with an ideal or nonenergetic multiport component $\mathcal{N}_i$ (a *perfect coupler* [4]) which has algebraic terminal equations, and possessing (identically) zero instantaneous power. With this representation the whole system becomes as a collection of several one-port components and a number of ideal multiport components. However, it is always possible to lump all of the multiport ideal components into a single equivalent multiport ideal component, $\mathcal{N}$ [12],[13].

In passive networks, a perfect coupler is either a multiport ideal transformer or a gyrator [14],[15]. In mechanical systems ideal rigid bodies behave like a multiport ideal transformer. However, the components corresponding to a multiport gyrator can also be observed if the operator matrix $\mathbf{W}(\frac{d}{dt}) = \mathbf{P}\frac{d}{dt} + \mathbf{Q}$ appearing in the terminal equations of the rigid body is further represented (synthesized) in terms of simpler components [17]. Indeed, consider the terminal equations of an $(n+1)$-port rigid body with terminals $(O, A_0, A_1, A_2, \cdots, A_n)$ corresponding to the terminal graph in Figure 1a, as given in equation (42) of [2], where the terminal A_0 is selected as the center of mass, G, of the rigid body :

$$\begin{matrix}(6n)\\(6)\end{matrix}\quad \begin{bmatrix}\mathbf{x}(t)\\ \mathbf{y}_G(t)\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{K}_G^T\\ -\mathbf{K}_G & \mathbf{W}_G(\frac{d}{dt})\end{bmatrix}\begin{bmatrix}\mathbf{y}(t)\\ \mathbf{x}_G(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{u}_G(t)\end{bmatrix} \tag{1}$$

with

$$\left.\begin{aligned}
\mathbf{x}_G &= \begin{bmatrix}\mathbf{v}_G\\ \boldsymbol{\omega}_G\end{bmatrix},\ \mathbf{y}_G = \begin{bmatrix}\mathbf{f}_G\\ \mathbf{M}_G\end{bmatrix}\\
\mathbf{K}_G &= [\mathbf{K}_{G1}\cdots\mathbf{K}_{Gn}],\ \mathbf{K}_{Gi} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{R}_{Gi} & \mathbf{I}\end{bmatrix},\ \mathbf{u}_G = \begin{bmatrix}-m\mathbf{g}\\ \mathbf{0}\end{bmatrix}\\
\mathbf{W}_G &= \mathbf{P}_G\tfrac{d}{dt} + \mathbf{Q}_G\\
\mathbf{P}_G &= \begin{bmatrix}m\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_G\end{bmatrix},\ \mathbf{Q}_G = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Omega}_G\mathbf{J}_G\end{bmatrix}
\end{aligned}\right\} \tag{2}$$

where $\mathbf{I}$ is the third order identity matrix, $\mathbf{R}_{Gi}$ is the 3×3 skew symmetric matrix corresponding to the position vector of A_i with respect to the mass center, G, and $\boldsymbol{\Omega}_G$ is also the skew symmetric matrix representation of the angular velocity vector, $\boldsymbol{\omega}_G$. In general, the inertia matrix $\mathbf{J}_G$ is nondiagonal. The schematic diagram of the rigid body is also shown in Figure 1b. Equation (1) will not have submatrices $\mathbf{W}_G(\frac{d}{dt})$ and $\mathbf{u}_G$ if a (scalar) 6-port component $\mathcal{W}_G$ with the terminal equations

$$\mathbf{y} = \mathbf{W}_G\mathbf{x} + \mathbf{u}_G \tag{3}$$

is (conceptually) extracted form the port (O, G) of the rigid body leaving it as an ideal component. This process is shown in Figure 1c. One may also perform a similar operation, this time extracting a different 6-port component $\mathcal{W}_{A_i}$ from the port (O, A_i) to have an equivalent representation for the rigid body [3]. At this stage, however, considering the expressions of $\mathbf{W}_G(\frac{d}{dt})$ in equation (2), we may represent the component $\mathcal{W}_G$ as two disjoint 3-port components, $(\mathcal{M})$ and $(\mathcal{J}, \mathcal{S})$ with the respective terminal equations of the form

$$(\mathcal{M}) \ :\ \mathbf{f}(t) = m\mathbf{I}\frac{d}{dt}\mathbf{v}(t) + \mathbf{u}_G \tag{4}$$

$$(\mathcal{J}, \mathcal{S}) \ :\ \mathbf{M}(t) = \left[\mathbf{J}_G\frac{d}{dt} + \boldsymbol{\Omega}\mathbf{J}_G\right]\boldsymbol{\omega} \tag{5}$$

as shown in Figure 1d. Furthermore, 3-port component $(\mathcal{J}, \mathcal{S})$ (representing the Euler's equation) can be considered as the parallel interconnection of two simpler 3-port components $(\mathcal{J})$ and $(\mathcal{S})$ with the respective terminal equations

$$(\mathcal{J}) \ :\ \mathbf{M}(t) = \mathbf{J}_G\frac{d}{dt}\boldsymbol{\omega}(t) \tag{6}$$

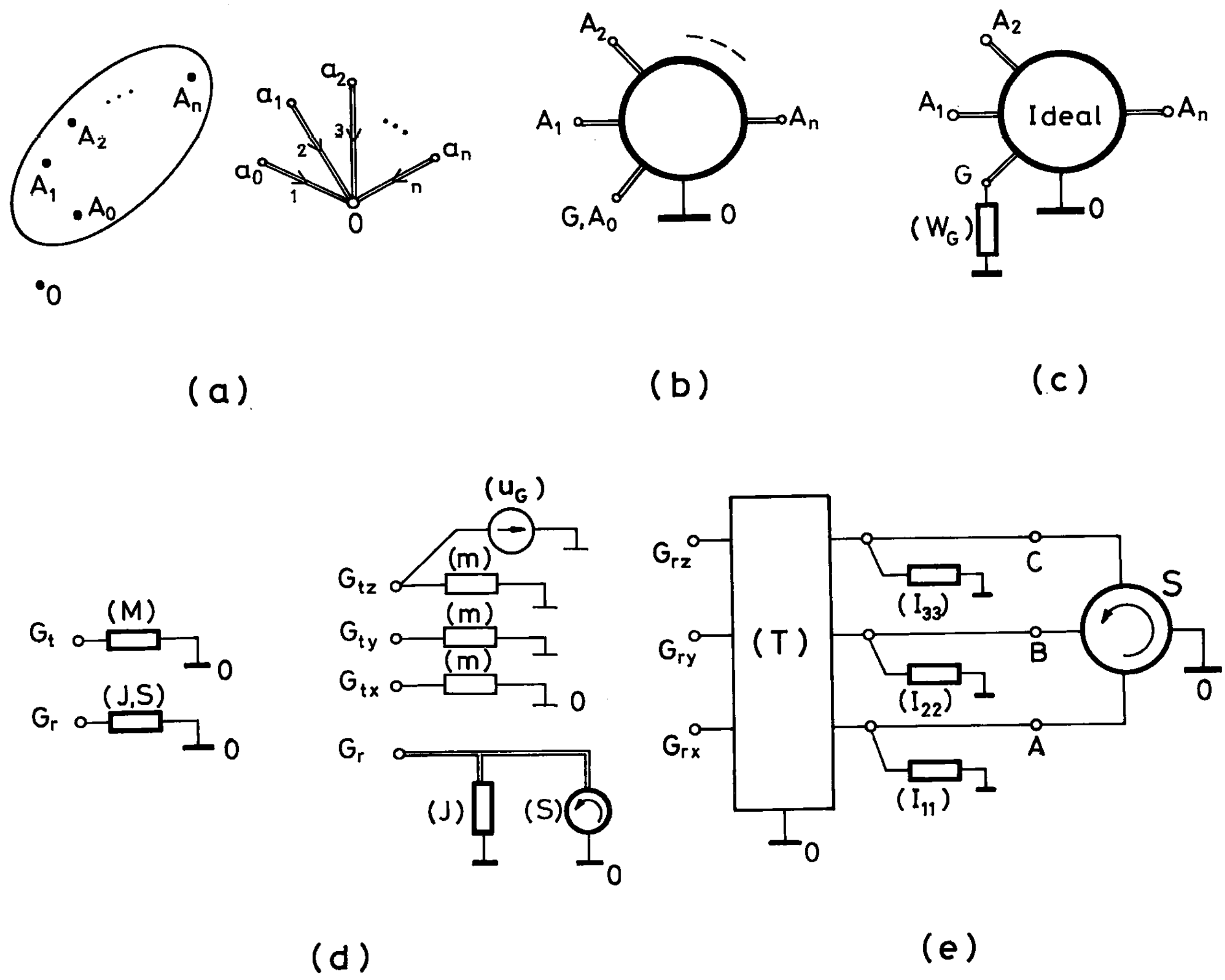
$$(\mathcal{S}) \ :\ \mathbf{M}(t) = \boldsymbol{\Omega}\mathbf{J}_G\,\boldsymbol{\omega}(t) \tag{7}$$

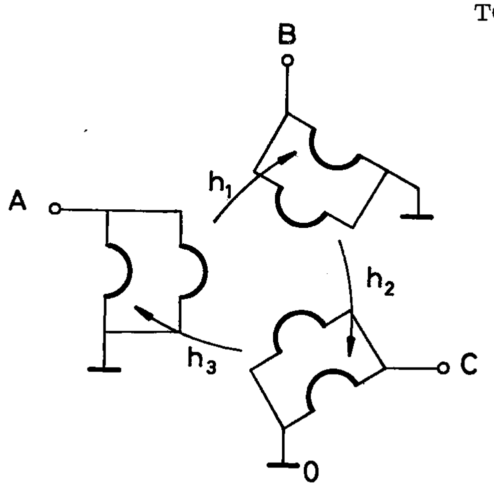
For further simplification of these representations, we may apply a coordinate transformation to equations (6) and (7), diagonalizing the symmetric and positive definite inertia matrix $\mathbf{J}_G$. This transformation can be interpreted as an ideal (transformer) 6-port component terminated in two 3-ports having similar terminal equations as in (6) and (7) except now the matrix $\mathbf{J}_G = \mathbf{J} = diag\,[I_{11}, I_{22}, I_{33}]$ is diagonal, and their representations are given in Figure 1e. The 3-port ($\mathcal{S}$) is an ideal (non energic) component (a circulator [21]), since its instantaneous power vanishes identically :

$$\wp(t) = \boldsymbol{\omega}^T(t)M(t) = \boldsymbol{\omega}^T(\Omega\mathbf{J})\boldsymbol{\omega} \equiv 0 \tag{8}$$

Hence, ($\mathcal{S}$) can be represented by the interconnected three 2-port gyrators as given in Figure 1f (see appendix).

It is clear from the above discussion that, a rigid body, as a multiport component, can be represented as an interconnected three 1-port (2-terminal) masses, three 1-port inertias, two ideal (transformers) components, one corresponding to the ideal rigid body, the other corresponding to the coordinate transformation and three 2-port gyrators. However, since we will be considering those mechanical systems in which the rigid bodies are in planar motion, the submatrix $\mathbf{Q}_G$ in equation (2) vanishes and the representations of rigid bodies will not contain the gyrators. In what follows, we shall state only those properties of ideal components corresponding to ideal transformers [13][16]. Other properties concerning the gyrators can be found elsewhere [18],[19].





(f)

Figure 1. (a) A rigid body as an $(n+1)-$port with terminals $(O, A_0, A_1, A_2, \cdots, A_n)$ and its (vector) terminal graph (a star-like or Lagrangian tree). (b) Schematic diagram of the rigid body. (c) A rigid body as a multiport component is represented (synthesized) as an ideal rigid body loaded by the one-port (scalar six-port) mass component $\mathcal{W}_G$ at the port (O, G). (d) Representing 6-port component $\mathcal{W}_G$ as two disjoint 3-port components $(\mathcal{M})$ (translational) and $(\mathcal{J}, \mathcal{S})$ (rotational). (e) Realization of 3-port components $(\mathcal{J})$ and $(\mathcal{S})$. Where $\mathbf{T}$ is an orthogonal transformation matrix corresponding to the turns ratio matrix of the ideal 6-port transformer $(\mathcal{T})$. (f) Realization of 3-port component $(\mathcal{S})$ by three 2-port gyrators.

- **Theorem - 1 :** Let $\mathcal{N}$ be an $n-$port ideal component (transformer) whose k_1-ports are connected rigidly (short circuited) to the reference and whose k_2-ports are left free (open circuited). Then the resulting $(n-k_1-k_2)-$port component is also ideal (transformer).

- **Theorem - 2 :** Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two n_1-port and n_2-port ideal components (transformers), respectively. A multiport component $\mathcal{N}$ obtained from an arbitrary interconnection scheme of $\mathcal{N}_1$ and $\mathcal{N}_2$ is also ideal.

Theorem-2 can be extended into the interconnection of more than two multiport ideal components by noticing that, after connecting any two ideal components to obtain a new multiport ideal component, $\mathcal{N}_{12}$, a third one may be connected to $\mathcal{N}_{12}$ to yield a new such one, $\mathcal{N}_{123}$. Therefore, the process of interconnecting one multiport ideal component at a time, finally results in a single ideal component.

Consideration of interconnected ideal components as a single ideal component, generally, allows one to eliminate automatically some of the terminal variables associated with the ideal components $\mathcal{N}_i$, and also permits one to consider only those terminal variables at the ports of $\mathcal{N}$ where the two-terminal components are to be connected. It is known that [9], since some of the internal variables of $\mathcal{N}$ may not be solvable, if the ideal components $\mathcal{N}_i$ are not lumped into a single ideal component $\mathcal{N}$, one may have the difficulties in eliminating unwanted variables during the process of deriving the state equations for that system. However, even if all ideal multiport components are lumped into a single equivalent ideal multiport $\mathcal{N}$, the terminal equations for $\mathcal{N}$ will exist in several possible forms each corresponding to the choice of different set of state variables, nevertheless some difficulties still persist in the formulation process : Part of the possible candidate set of state variables may not all appear in the final set of state equations and their nonexistence bring the final set of state equations into a more complicated form. At this stage, topological methods give the information as to which state variables will disappear from a candidate set [20]. Elimination of least number of state variables in a candidate set is preferable and it can be accomplished by the consideration of carefully chosen mathematical model amongst the available models of $\mathcal{N}$. With this choice, although the number of state equations are now increased, however the expression of each equation becomes less complex.

The selection of a proper tree, T, in the system graph requires the following hierarchy [6],[22],[23]:

- (a) Edges corresponding to the across (velocity) drivers are included in T. (Otherwise the system is inconsistent i.e., these edges do not form a circuit in the system graph).
- (b) If the terminal equations of a multi-port ideal component are of the form

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{K}^T \\ -\mathbf{K} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{9}$$

  then the edges corresponding to the across variables in $\mathbf{x}_1$ are included in T while those edges corresponding to the through variables in $\mathbf{y}_2$ are included in the co-tree T' of T.
- (c) Edges corresponding to the two-terminal mass and inertia components must be in T. However some of such edges may not be included in T if they form a circuit among themselves or also including the edges corresponding to the across drivers.
- (d) At this stage, if a proper tree T has not been obtained yet, then some edges corresponding to the damper and perhaps to the spring components are added to complete it.

With this hierarchical process all the edges corresponding to the through (force or torque) drivers remain in the co-tree T'. (Otherwise the system becomes inconsistent i.e., these edges do not form a cut-set in the system graph).

Note that the terminal equations of an ideal component given in (9), depending upon the rank of $\mathbf{K}$, can be written in different forms. This implies that selection of the proper tree is not unique, hence there is no unique form for the state equations. Note also that to simplify the formulation procedure, we may as well use only the relation $\mathbf{x}_1 = \mathbf{K}^T\mathbf{x}_2$ in equation (9) since the modification of this relation as $\mathbf{x}_1' = (\mathbf{K}')^T\mathbf{x}_2'$ will automatically yield a modified relation among the through variables as $\mathbf{y}_2' = -\mathbf{K}'\mathbf{y}_1'$. All of these properties are illustrated in section 4.

The state equations of the mechanical system will have the following general form :

$$\frac{d}{dt}\mathbf{z} = \mathbf{f}(\mathbf{z}, \mathbf{u}) \tag{10}$$

where the vector $\mathbf{u}$ corresponds to the known driving functions while the state vector $\mathbf{z}$ contains the across variables (translational or angular velocities or both) associated with the rigid bodies and also the through variable (forces or torques or both) of the spring components. If these across and through variables are indicated separately by the vectors $\mathbf{x}$ and $\mathbf{y}$, respectively, then equation (10) can be written as

$$\frac{d}{dt}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1(\mathbf{x}, \mathbf{y}, \mathbf{u}) \\ \mathbf{f}_2(\mathbf{x}, \mathbf{y}, \mathbf{u}) \end{bmatrix} \tag{11}$$

In mechanical systems, however, the functions $\mathbf{f}_1$ and $\mathbf{f}_2$ may also contain the variables corresponding to the integrals of the across and through variables. i.e., the displacements (translational or rotational or both) $\mathbf{d}$ and the momenta (linear or angular or both) $\mathbf{p}$. In particular, these variables necessarily appear in the terminal equations of multiport ideal components. Therefore, the state equations in (11) must be augmented to include these integral variables as well, yielding

$$\frac{d}{dt}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{d} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1(\mathbf{x},\mathbf{y},\mathbf{d},\mathbf{p},\mathbf{u}) \\ \mathbf{f}_2(\mathbf{x},\mathbf{y},\mathbf{d},\mathbf{p},\mathbf{u}) \\ \mathbf{x} \\ \mathbf{y} \end{bmatrix} \tag{12}$$

which is also in the form of equation (10). Note that the construction procedure of equation (12) from that in (11) may not be staightforward. Indeed, any one of the variables in $\mathbf{d}$ or $\mathbf{p}$ may be the integral of an across or a through variable which has already been eliminated during the process and it does not appear in $\mathbf{x}$ or $\mathbf{y}$ of equation (11). If this is the case, further augmentation of the state equations is needed to have its complete form. However, this second augmentation step may be avoided by carefully selecting the form of the terminal equations of the ideal component $\mathcal{N}$ at the begin with.

3. Topological Conditions on Ideal Rigid Bodies

In mechanical systems ideal components result from the idealization of rigid bodies. The schematic representation of such an ideal component as $(n+1)$-port with terminals $(O, A_0, A_1, A_2, \cdots, A_n)$ corresponding to the vector terminal graph is given in Figure 1a. It has the terminal equations [2]

$$\begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_n(t) \\ \mathbf{y}_0(t) \end{bmatrix} = \begin{bmatrix} & & 0 & & (\mathbf{K}_1)^T \\ & & & & (\mathbf{K}_2)^T \\ & & & & \vdots \\ & & & & (\mathbf{K}_n)^T \\ -\mathbf{K}_1 & -\mathbf{K}_2 & \cdots & -\mathbf{K}_n & \end{bmatrix} \begin{bmatrix} \mathbf{y}_1(t) \\ \mathbf{y}_2(t) \\ \vdots \\ \mathbf{y}_n(t) \\ \mathbf{x}_0(t) \end{bmatrix} \tag{13}$$

which is of the form of equation (9) and where the 6×6 submatrices $\mathbf{K}_i$ are nonsingular. Therefore, any one of the variables $\mathbf{y}_i(t)$ can be transferred to the other side of equation (13). i.e., in all possible forms of these equivalent terminal equations, the across variables associated with only one port of the ideal component can be specified arbitrarily. According to the topological conditions stated for equation (9), in the terminal graph of the ideal component all the ports except an arbitrarily selected one must be included into a tree. However, not all six variables in $\mathbf{y}_i$ but a subset of it may also be transferred to the other side. This situation becomes important when the port variables are considered in the component (scalar) level. In planar motion this may be exactly the case. Hence, in general, the topological conditions must be tested based on the scalar terminal graph.

Note that, in planar motion the across and through port variables simplify into [3]

$$\mathbf{x}_i = \begin{bmatrix} v_{ix} \\ v_{iy} \\ \omega_{iz} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_i = \begin{bmatrix} f_{ix} \\ f_{iy} \\ M_{iz} \end{bmatrix} \qquad (i = 0, 1, \cdots, n) \tag{14}$$

and the expression of the submatrices $\mathbf{K}_i$ takes the following form

$$\mathbf{K}_i = \begin{bmatrix} 1 & & \\ & 1 & \\ -r_{0iy} & r_{0ix} & 1 \end{bmatrix} \tag{15}$$

where $\mathbf{r}_{0i} = [r_{0ix} \; r_{0iy} \; 0]^T$ represents the position vector of the point A_i with respect to the point A_0 on the rigid body. In application, A_0 is taken as the center of mass G of the rigid body since under only this choice the terminal equations take their simplest form.

4. Examples

Example - 1 *A Rocking Pendulum* [24] shown in Figure 2a is considered as a two-port rigid body with terminals (G, P, O). Here the terminal P does not correspond to a fixed point of the rigid body, its position changes during the motion. It is assumed that there is no slipping at P i.e., the motion of pendulum is pure rolling. It may be of some interest to determine the reaction force $\mathbf{f}_P$ at this point. The position vector $\mathbf{r}_{GP}$ of P with respect to the mass center G is

$$\mathbf{r}_{GP} = \begin{bmatrix} r_x \\ r_y \end{bmatrix} = \begin{bmatrix} -\ell \sin\theta \\ \ell\cos\theta - r \end{bmatrix} \tag{16}$$

- (i) *Direct Formulation* - The terminal equations of the two-port rigid body in planar motion are [3]

$$\begin{bmatrix} \mathbf{x}_P \\ \mathbf{y}_O \end{bmatrix} = \begin{bmatrix} & \mathbf{K}^T \\ -\mathbf{K} & \mathbf{W}_G(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} \mathbf{y}_P \\ \mathbf{x}_G \end{bmatrix} + \begin{bmatrix} \\ \mathbf{u}_G \end{bmatrix} \tag{17}$$

where $\mathbf{W}_G(\frac{d}{dt}) = \mathbf{P}_G \frac{d}{dt} + \mathbf{Q}_G$ with

$$\mathbf{P}_G = \begin{bmatrix} m & & \\ & m & \\ & & J_G \end{bmatrix}, \quad \mathbf{Q}_G = \mathbf{0} \quad \text{and} \quad \mathbf{u}_G = -m\mathbf{g} = \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix} \tag{18}$$

Let k be the radius of gyration i.e., $J_G = mk^2$. The terminal conditions observed from Figure 2a are: $f_{Gx} = f_{Gy} = M_{Gz} = 0$ (no connection is made at port OG), $v_{Px} = v_{Py} = 0$ (P is the instantaneous rotation center of the pendulum) and $M_{Pz} = 0$. With these terminal conditions equation (17) takes the following explicit form

$$\begin{bmatrix} 0 \\ 0 \\ \omega_P \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} & & & 1 & & -r_y \\ & & 0 & & 1 & r_x \\ & & & & & 1 \\ -1 & & & m\frac{d}{dt} & & \\ & -1 & & & m\frac{d}{dt} & \\ r_y & -r_x & -1 & & & J_G\frac{d}{dt} \end{bmatrix} \begin{bmatrix} f_{Px} \\ f_{Py} \\ 0 \\ v_{Gx} \\ v_{Gy} \\ \omega_G \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ mg \\ 0 \end{bmatrix} \tag{19}$$

In equation (19), the third row gives $\omega_P = \omega_G = \omega$. From the first two rows and equation (16), we have

$$\begin{bmatrix} v_{Gx} \\ v_{Gy} \end{bmatrix} = \begin{bmatrix} \ell\cos\theta - r \\ \ell\sin\theta \end{bmatrix} \omega_G \tag{20}$$

therefore,

$$\frac{d}{dt}\begin{bmatrix} v_{Gx} \\ v_{Gy} \end{bmatrix} = \begin{bmatrix} \ell\cos\theta - r \\ \ell\sin\theta \end{bmatrix} \frac{d}{dt}\omega + \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \ell\,\omega^2 \tag{21}$$

In equation (19), fourth and fifth rows determine the reaction force at P :

$$\begin{bmatrix} f_{Px} \\ f_{Py} \end{bmatrix} = m\frac{d}{dt}\begin{bmatrix} v_{Gx} \\ v_{Gy} \end{bmatrix} + \begin{bmatrix} 0 \\ mg \end{bmatrix}$$

or considering equation (21), we obtain

$$\begin{bmatrix} f_{Px} \\ f_{Py} \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} m\ell\omega^2 + \begin{bmatrix} \ell\cos\theta - r \\ \ell\sin\theta \end{bmatrix} m\frac{d}{dt}\omega + \begin{bmatrix} 0 \\ mg \end{bmatrix} \tag{22}$$

Equation of motion for the rocking pendulum is obtained from the last row of equation (19) :

$$[\,\ell\cos\theta - r \quad \ell\sin\theta]\begin{bmatrix} f_{Px} \\ f_{Py} \end{bmatrix} + J_G\frac{d}{dt}\omega = 0 \tag{23}$$

Since $J_G = mk^2$, substituting equation (22) into equation (23) gives

$$[(\ell\cos\theta - r)^2 + \ell^2\sin^2\theta + k^2]\,\frac{d}{dt}\omega + \ell\,[-(\ell\cos\theta - r)\sin\theta + \ell\sin\theta\cos\theta]\,\omega^2 + g\ell\sin\theta = 0$$

or finally

$$\frac{d}{dt}\omega = -\frac{\ell\sin\theta(g + r\omega^2)}{\ell^2 + r^2 + k^2 - 2\ell r\cos\theta} = F(\omega,\,\theta) \tag{24}$$

Since $\omega = \frac{d}{dt}\theta$, we obtain the equations of motion in the state model form :

$$\frac{d}{dt}\begin{bmatrix} \omega \\ \theta \end{bmatrix} = \begin{bmatrix} F(\omega,\,\theta) \\ \omega \end{bmatrix} \tag{25}$$

- (ii) *State Model Formulation* - the schematic diagram of the pendulum is represented as in Figure 2b together with the terminal conditions. This system contains only one ideal component. The *system graph* is also given in Figure 2d. Due to the terminal conditions ($v_{Px} \equiv 0, v_{Py} \equiv 0$ and $M_{Pz} \equiv 0$), the edges $(p_x, 0)$ and $(p_y, 0)$ in the terminal graph of the ideal component must be chords while the edge $(p_z, 0)$ must be a branch. This in turn implies that the terminal equations of the ideal pendulum as a two-port (or scalar six-port) component cannot be taken in the form :

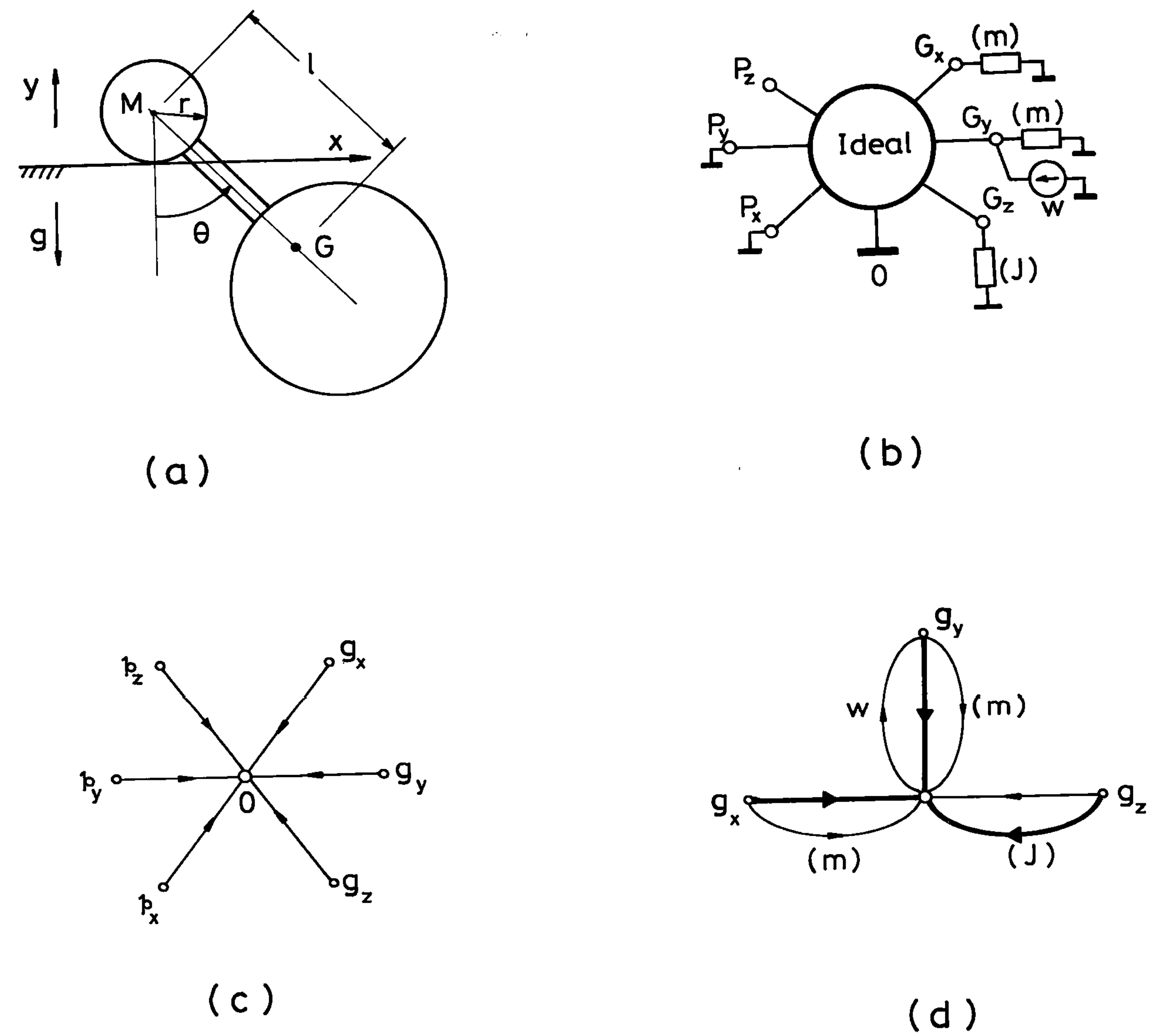


Figure - 2 (a) A rocking pendulum. P^{-} **is the instantaneous** rotation center, G is the mass center of the pendulum. (b) Schematic diagram of the pendulum endowed with the terminal conditions. (c) The terminal graph of the ideal pendulum. (d) A proper formulation tree in the system graph. The edges (branches) of the tree are indicated by heavy lines.

$$\begin{bmatrix} v_{Px} \\ v_{Py} \\ \omega_{Pz} \\ \\ f_{Gx} \\ f_{Gy} \\ M_{Gx} \end{bmatrix} = \begin{bmatrix} & & & 1 & & -r_y \\ & & 0 & & 1 & r_x \\ & & & & & 1 \\ \\ -1 & & & & & \\ & -1 & & & & \\ r_y & -r_x & -1 & & & \end{bmatrix} \begin{bmatrix} f_{Px} \\ f_{Py} \\ M_{Pz} \\ \\ v_{Gx} \\ v_{Gy} \\ \omega_{Gz} \end{bmatrix} \tag{26}$$

These terminal equations must be modified to meet the topological condition as

$$\begin{bmatrix} f_{Px} \\ f_{Py} \\ \omega_{Pz} \\ \\ v_{Gx} \\ v_{Gy} \\ M_{Gz} \end{bmatrix} = \begin{bmatrix} & & & -1 & & \\ & & & & -1 & \\ & & 0 & & & 1 \\ & & & & & \\ 1 & & & & & r_y \\ & 1 & & & & -r_x \\ & & -1 & -r_y & r_x & \end{bmatrix} \begin{bmatrix} v_{Px} \\ v_{Py} \\ M_{Pz} \\ \\ f_{Gx} \\ f_{Gy} \\ \omega_{Gz} \end{bmatrix} \tag{27}$$

On the other hand, considering the terminal conditions ($v_{Px} = v_{Py} \equiv 0,\ M_{Pz} \equiv 0$) and omitting the variables associated with the port (PO), the ideal pendulum can be represented as a tree-port component with terminals (O, G_x, G_y, G_z) having the following terminal equations corresponding to the terminal graph in Figure 2c :

$$\begin{bmatrix} v_{Gx} \\ v_{Gy} \\ M_{Gz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & r_y \\ 0 & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \begin{bmatrix} f_{Gx} \\ f_{Gy} \\ \omega_{Gz} \end{bmatrix} \tag{28}$$

This form of the terminal equations implies that the edges $(g_x, 0)$ and $(g_y, 0)$ must be branches while the edge $(g_z, 0)$ must be a chord. These observations determines a tree in the system graph as shown by heavy lines in Figure 2d. From the system graph we see that only the edge corresponding to the inertia component (J) is included into the formulation tree. i.e., the system has only one state variable ω_J. The terminal equations of the inertia and the mass components are in the form:

$$J\frac{d}{dt}\omega_J = M_J \tag{29}$$

$$m\frac{d}{dt}v_{mx} = f_{mx} \tag{30}$$

$$m\frac{d}{dt}v_{my} = f_{my} \tag{31}$$

With the terminal equations in (28)-(31) and also the circuit and cut-set equations implied by the system graph in Figure 2d, the state equation can be obtained through a systematic substitution process: The rough form of the state equation is as in (29). Therefore the variable M_J must be expressed in terms of the state variable ω_J and the driving function $w = -mg$. From the system graph $M_J = -M_{Gz}$. But the last equation in (28) gives

$$M_J = -M_{Gz} = -\begin{bmatrix} -r_y & r_x \end{bmatrix} \begin{bmatrix} f_{Gx} \\ f_{Gy} \end{bmatrix} \tag{32}$$

$(M_{Pz} = -M_z \equiv 0)$. On the other hand, again from the system graph and equations (30) and (31) we have

$$\begin{bmatrix} f_{Gx} \\ f_{Gy} \end{bmatrix} = \begin{bmatrix} -f_{mx} \\ -f_{my} + w \end{bmatrix} = -m\frac{d}{dt}\begin{bmatrix} v_{mx} \\ v_{my} \end{bmatrix} + \begin{bmatrix} 0 \\ -mg \end{bmatrix} \tag{33}$$

Further, from the system graph and equation (28)

$$\begin{bmatrix} v_{mx} \\ v_{my} \end{bmatrix} = \begin{bmatrix} v_{Gx} \\ v_{Gy} \end{bmatrix} = \begin{bmatrix} r_y \\ -r_x \end{bmatrix} \omega_{Gz} \tag{34}$$

Substituting equation (34) into equation (33) and that into (32) yields

$$\left.\begin{array}{rcl} M_J & = & -[-r_y \ \ r_x]\left\{-m\frac{d}{dt}\left(\begin{bmatrix} r_y \\ -r_x \end{bmatrix}\omega_J\right)-\begin{bmatrix} 0 \\ mg \end{bmatrix}\right\} \\ & = & -m[r_y \ \ -r_x]\begin{bmatrix} r_y \\ -r_x \end{bmatrix}\frac{d}{dt}\omega_J - m\omega_J[r_y \ \ -r_x]\frac{d}{dt}\begin{bmatrix} r_y \\ -r_x \end{bmatrix} + r_x mg \\ & = & -m(r_x^2+r_y^2)\frac{d}{dt}\omega_J - \frac{1}{2}m\omega_J\frac{d}{dt}(r_x^2+r_y^2)+r_x mg \end{array}\right\} \quad (35)$$

Explicit expression of r_x and r_y are given in equation (16). Therefore, $r_x^2+r_y^2=\ell^2+r^2-2r\ell\cos\theta$, $\frac{d}{dt}(r_x^2+r_y^2)=2r\ell\sin\theta\,\omega_J$. Substituting equation (35) into (29) and considering the fact that $J=mk^2$, we arrive at equation (24) and finally at the augmented state equations in (25) where ω_J is replaced by ω.

Example - 2 : In a vertical plane two rigid homogeneous discs $\mathcal{R}_1$ and $\mathcal{R}_2$ of respective radii b and a ($\mathcal{R}_2$ is rolling inside $\mathcal{R}_1$ without slipping) are joined by a bearing part $\mathcal{R}$. Disc $\mathcal{R}_1$ also rolls without skidding on a horizontal plane as shown in Figure 3a. With a given arbitrary initial positions and velocities for $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}$, due to the gravity forces, the mechanical system will oscillate. Assume viscous frictions $(\mathcal{B}_a)$ and $(\mathcal{B}_b)$ at the bearings (A, G_1) and (B, G_2), respectively. The masses and the inertias with respect to the mass centers G_1, G_2, G, of $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}$ are also given as m_1, m_2, m, J_1, J_2 and J, respectively. The state equations of the system will be obtained.

The mechanical system in Figure 3a contains three multiterminal rigid bodies $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}$ and two two-terminal dampers $(\mathcal{B}_a)$, $(\mathcal{B}_b)$. The interconnection pattern and the terminal conditions of the components are shown in the schematic diagram of Figure 3b. The terminal equations of the rigid bodies are given in the following forms corresponding to a star-like (Lagrangian) tree terminal graphs [6],[8], :

$$(\mathcal{R}_1): \begin{bmatrix} \mathbf{x}_{P1} \\ \mathbf{x}_{P2} \\ \mathbf{y}_{G1} \end{bmatrix} = \begin{bmatrix} & 0 & \mathbf{K}_1^T \\ & & \mathbf{K}_2^T \\ -\mathbf{K}_1 & -\mathbf{K}_2 & \mathbf{W}_1 \end{bmatrix}\begin{bmatrix} \mathbf{y}_{P1} \\ \mathbf{y}_{P2} \\ \mathbf{x}_{G1} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{u}_1 \end{bmatrix}$$

$$with \left\{ \begin{array}{rcl} \mathbf{K}_1 & = & \begin{bmatrix} 1 & & \\ & 1 & \\ b & 0 & 1 \end{bmatrix}, \ \mathbf{u}_1 = -m_1\mathbf{g} \\ \mathbf{K}_2 & = & \begin{bmatrix} 1 & & \\ & 1 & \\ bC & bS & 1 \end{bmatrix} \end{array}\right. \quad (36)$$

$$(\mathcal{R}_2): \begin{bmatrix} \mathbf{x}_{P3} \\ \mathbf{y}_{G2} \end{bmatrix} = \begin{bmatrix} & \mathbf{K}_3^T \\ -\mathbf{K}_3 & \mathbf{W}_2 \end{bmatrix}\begin{bmatrix} \mathbf{y}_{P3} \\ \mathbf{x}_{G2} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{u}_2 \end{bmatrix}$$

$$with \left\{ \begin{array}{rcl} \mathbf{K}_3 & = & \begin{bmatrix} 1 & & \\ & 1 & \\ aC & aS & 1 \end{bmatrix} \\ \mathbf{u}_2 & = & -m_2\mathbf{g} \end{array}\right. \quad (37)$$

$$(\mathcal{R}): \begin{bmatrix} \mathbf{x}_{A} \\ \mathbf{x}_{B} \\ \mathbf{y}_{G} \end{bmatrix} = \begin{bmatrix} & 0 & \mathbf{K}_A^T \\ & & \mathbf{K}_B^T \\ -\mathbf{K}_A & -\mathbf{K}_B & \mathbf{W} \end{bmatrix}\begin{bmatrix} \mathbf{y}_{A} \\ \mathbf{y}_{B} \\ \mathbf{x}_{G} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{u} \end{bmatrix}$$

$$
\text{with} \left\{ \begin{array}{lcl} \mathbf{K}_A & = & \begin{bmatrix} 1 & & \\ & 1 & \\ -dC & -dS & 1 \end{bmatrix}, \quad \mathbf{u} = -m\mathbf{g} \\ \mathbf{K}_B & = & \begin{bmatrix} 1 & & \\ & 1 & \\ (c-d)C & (c-d)S & 1 \end{bmatrix} \end{array} \right. \tag{38}
$$

where the symbols C and S stand for $\cos\phi$ and $\sin\phi$, respectively. The diagonal operator matrices $\mathbf{W}_1(\frac{d}{dt})$, $\mathbf{W}_2(\frac{d}{dt})$ and $\mathbf{W}(\frac{d}{dt})$ can be represented separately as two-terminal mass and inertia components, hence the rigid bodies become ideal components. Their terminal equations are exactly as in equations (36), (37) and (38) with the missing operator matrices. On the other hand the terminal equations of the dampers are

$$
\left. \begin{array}{lcl} M_a & = & B_a\,\omega_a \\ M_b & = & B_b\,\omega_b \end{array} \right\} \tag{39}
$$

From Figure 3a, at $t > 0$, the angles ϕ, θ and ψ are all positive. Their derivatives gives the angular velocities :

$$
\frac{d}{dt}\phi = \omega_{Gz} \triangleq \omega \;\; , \;\; \frac{d}{dt}\theta = \omega_{G1z} \triangleq \omega_1 \;\; , \;\; \frac{d}{dt}\psi = \omega_{G2z} \triangleq \omega_2 \tag{40}
$$

The equality of the arc lengths of the circles yields $a(\psi - \phi) = b(\theta - \phi)$. Since $c = (b - a)$, then we have

$$
\phi = \left(\frac{b}{c}\right)\theta - \left(\frac{a}{c}\right)\psi \tag{41}
$$

and the time derivative of both sides yields

$$
\omega = \left(\frac{b}{c}\right)\omega_1 - \left(\frac{a}{c}\right)\omega_2 \tag{42}
$$

As the first step, we shall lump all of these ideal components into the form of one multiport ideal component having only those terminals at which two-terminal mass, inertia and damping components are to be connected. To simplify the formulation procedure, first the ideal components $\mathcal{R}_1$ and $\mathcal{R}_2$ in Figure 3b will be lumped into a single ideal component, $\mathcal{R}_{12}$, having only the terminals $(O, G_{1x}, G_{1y}, G_{1z}, G_{2x}, G_{2y}, G_{2z})$ where the two-terminal components and the terminals of $\mathcal{R}$ will be connected. Considering only the across variables in equations (36) and (37), from the terminal and interconnection conditions of the ideal components $\mathcal{R}_1$ and $\mathcal{R}_2$, we have

$$
\begin{bmatrix} v_{P1x} \\ v_{P1y} \\ \omega_{P1z} \\ \\ v_{P2x} \\ v_{P2y} \\ \omega_{P2z} \end{bmatrix} = \begin{bmatrix} 1 & & b \\ & 1 & \\ & & 1 \\ \\ 1 & & bC \\ & 1 & bS \\ & & 1 \end{bmatrix} \begin{bmatrix} v_{G1x} \\ v_{G1y} \\ \omega_{G1z} \end{bmatrix} \;, \quad \begin{bmatrix} v_{P3x} \\ v_{P3y} \\ \omega_{P3z} \end{bmatrix} = \begin{bmatrix} 1 & & aC \\ & 1 & aS \\ & & 1 \end{bmatrix} \begin{bmatrix} v_{G2x} \\ v_{G2y} \\ \omega_{G2z} \end{bmatrix} \tag{43}
$$

where $v_{P1x} = v_{P1y} \equiv 0$ (P_1 is the instantaneous rotation center of $\mathcal{R}_1$) and $v_{Px2} = v_{P3x}$, $v_{P2y} = v_{P3y}$ (circuit equations). Note also that other terminal conditions, i.e., $M_{P1z} = M_{P2z} = M_{P3z} \equiv 0$ imply the fact that we may omit in writing the obvious equations $\omega_{P1z} = \omega_{P2z} = \omega_{G1z} = \omega_1$ and $\omega_{P3z} = \omega_{G2z} = \omega_2$. Hence from equations (43) the terminal equations of the ideal component $\mathcal{R}_{12}$ corresponding to the terminal graph in Figure 3c, is obtained as

$$\begin{bmatrix} v_{G1x} \\ v_{G1y} \\ \\ v_{G2x} \\ v_{G2y} \end{bmatrix} = \begin{bmatrix} -b & 0 \\ 0 & 0 \\ \\ b(C-1) & -aC \\ bS & -aS \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \tag{44}$$

Next consider the interconnected ideal components $\mathcal{R}_{12}$ and $\mathcal{R}$ in Figure 3d.

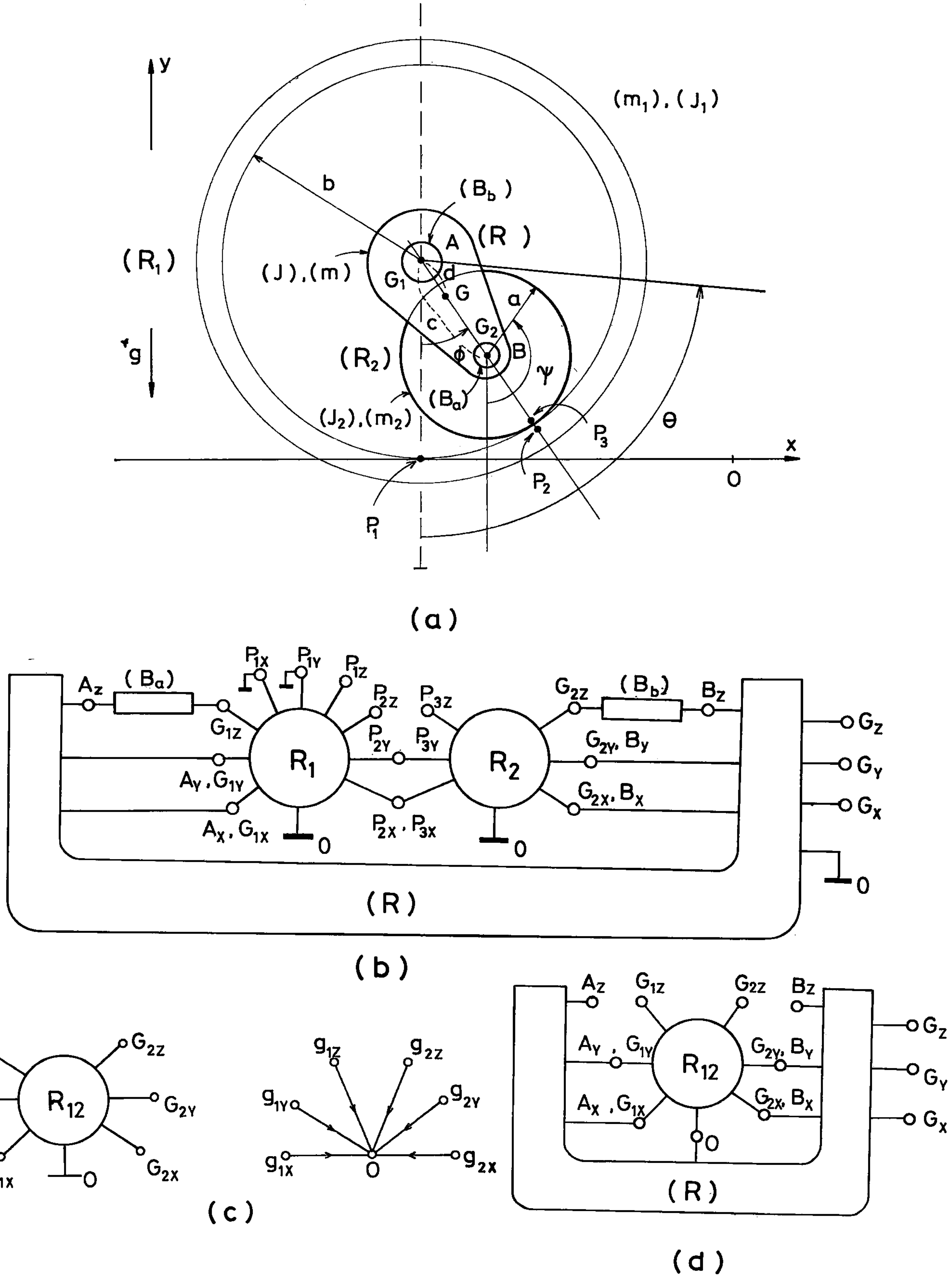
The equivalent ideal component $\mathcal{R}_{123}$ will contain the terminals $(O, G_{1x}, G_{1y}, G_{1z}, G_{2x}, G_{2y}, G_{2z}, A_z, B_z, G_x, G_y, G_z)$. Since the two-terminal masses, inertias and force drivers ($w_1 = -m_1 g$, $w_2 = -m_2 g$, $w = -mg$) are connected between the terminals $G_{1x}, G_{1y}, G_{1z}, G_{2x}, G_{2y}, G_{2z}, G_x, G_y, G_z$ and the reference terminal O, while the damping components $(\mathcal{B}_a)$, $(\mathcal{B}_b)$ are connected between the pairs of terminals (A_z, G_{1z}) and (B_z, G_{2z}), respectively, the terminal graph of $\mathcal{R}_{123}$ is selected as shown in Figure 3e. Initially, it is sufficient to consider only the velocity variables in the terminal equations of $\mathcal{R}_{123}$. In order to obtain these terminal equations from those of $\mathcal{R}_{12}$ in (44) and those of $\mathcal{R}$ in (38) which has the explicit form

$$\begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_B \end{bmatrix} = \begin{bmatrix} v_{Ax} \\ v_{Ay} \\ \omega_{Az} \\ \\ v_{Bx} \\ v_{By} \\ \omega_{Bz} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_A^T \\ \mathbf{K}_B^T \end{bmatrix} = \begin{bmatrix} 1 & & -dC \\ & 1 & -S \\ & & 1 \\ \\ 1 & & (c-d)C \\ & 1 & (c-d)S \\ & & 1 \end{bmatrix} \begin{bmatrix} v_{Gx} \\ v_{Gy} \\ \omega_{Gz} \end{bmatrix} , \quad (\omega_{Gz} = \omega) \tag{45}$$

we observe the interconnection conditions in Figure 3d, which yield $v_{Ax} = v_{G1x}, v_{Ay} = v_{G1y}, v_{Bx} = v_{G2x}$ and $v_{By} = v_{G2y}$ (circuit equations). Hence, from the first two rows in both equations (44) and (45)

$$\begin{bmatrix} -b & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} v_{Gx} \\ v_{Gy} \end{bmatrix} - d \begin{bmatrix} C \\ S \end{bmatrix} \omega \tag{46}$$

and from the remaining rows in equations (44) and (45)


y
(m1), (J1)
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(R1)
(J), (m)
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G1
d
G
a
c
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(R2)
ϕ
B
ψ
(Ba)
(J2), (m2)
P3
θ
x
P2
0
P1
(a)
Az
(Ba)
P1X
P1Y
P1Z
P2Z
P3Z
G2Z
(Bb)
BZ
G1Z
R1
P2Y
P3Y
R2
G2Y, BY
GZ
GY
AY, G1Y
G2X, BX
GX
AX, G1X
0
P2X, P3X
0
0
(R)
(b)
G1Z
G2Z
G1Y
R12
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G1X
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G2X
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(c)
AZ
G1Z
G2Z
BZ
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AY, G1Y
R12
G2Y, BY
GY
AX, G1X
G2X, BX
GX
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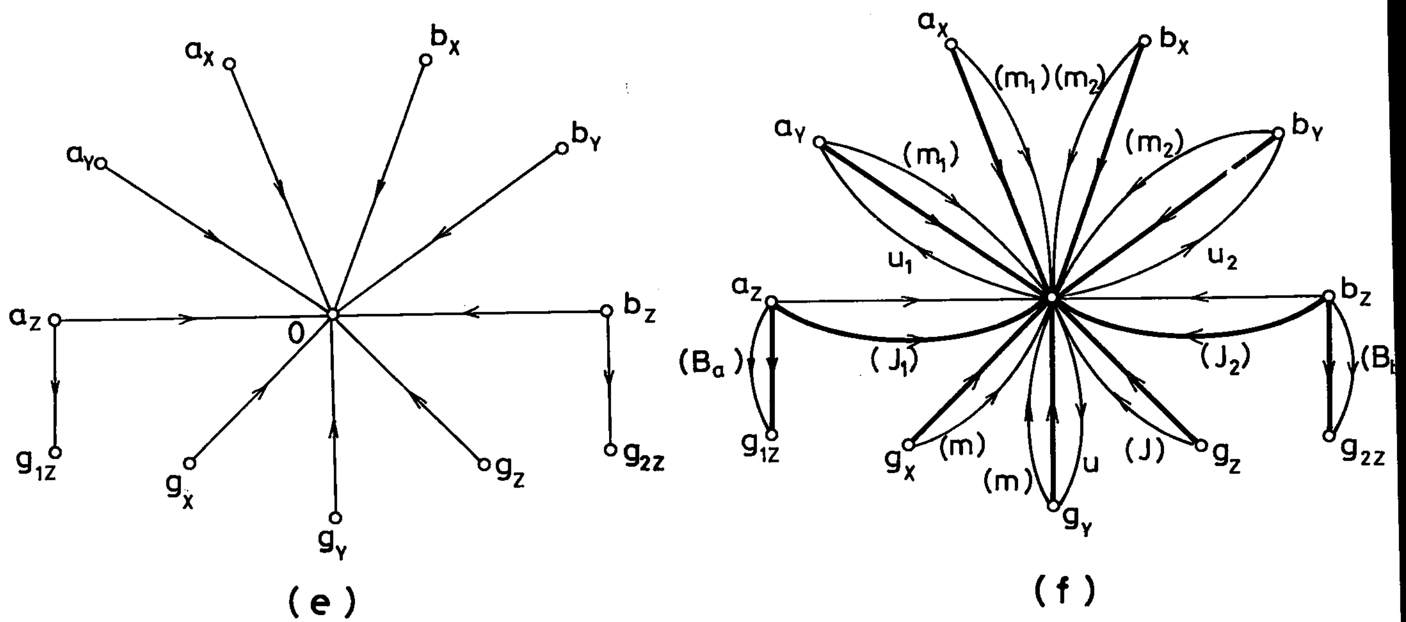

Figure - 3 (a) A mechanical system containing three rigid bodies $(\mathcal{R}_1), (\mathcal{R}_2), (\mathcal{R})$ and two rotational 2-port viscous friction components $(\mathcal{B}_a)$ and $(\mathcal{B}_b)$. (b) Schematic diagram of the system indicating the interconnection pattern of the components and the terminal conditions. (c) Schematic diagram and the terminal graph of the ideal component $(\mathcal{R}_{12})$ resulting from the interconnected idealized $(\mathcal{R}_1)$ and $(\mathcal{R}_2)$. (d) Schematic diagram of the ideal component $(\mathcal{R}_{123})$ obtained by interconnecting the ideal component $(\mathcal{R}_{12})$ and the idealized component $(\mathcal{R})$. (e) The terminal graph of the ideal component $(\mathcal{R}_{123})$. (f) The system graph and the selected proper tree.

$$\begin{bmatrix} b(C-1) & -aC \\ bS & -aS \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} v_{Gx} \\ v_{Gy} \end{bmatrix} + (c-d) \begin{bmatrix} C \\ S \end{bmatrix} \omega \tag{47}$$

Elimination of variables v_{Gx} and v_{Gy} from the equations (46) and (47) yields

$$b\omega_1 - a\omega_2 = c\omega \tag{48}$$

which is the relation already established in equation (42) from the geometrical considerations. Therefore, e.g., equations (46) and (48) give

$$\begin{bmatrix} v_{Gx} \\ v_{Gy} \end{bmatrix} = (1/c) \begin{bmatrix} b(dC-c) & -adC \\ bdS & adS \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \tag{49}$$

On the other hand, the circuit equations for the angular velocities ω_a and ω_b at the ports (A_z, G_{1z}) and (B_z, G_{2z}) together with equation (48), yield

$$\left.\begin{array}{lllllll} \omega_a & = & \omega_{Az} - \omega_{G1z} & = & \omega - \omega_1 & = & (a/c)(\omega_1 - \omega_2) \\ \omega_b & = & \omega_{Bz} - \omega_{G2z} & = & \omega - \omega_2 & = & (b/c)(\omega_1 - \omega_2) \end{array}\right\} \tag{50}$$

where the relations in equation (40) were used. Equation (44), (48), (49) and (50) give the velocity portion of the terminal equations of the ideal component $\mathcal{R}_{123}$ corresponding to the terminal graph in Figure 3e :

$$\mathbf{x}_1 = \begin{bmatrix} v_{G1x} \\ v_{G1y} \\ v_{G2x} \\ v_{G2y} \\ v_{Gx} \\ v_{Gy} \\ \\ \omega \\ \omega_a \\ \omega_b \end{bmatrix} = \begin{bmatrix} -b & 0 \\ 0 & 0 \\ b(C-1) & -aC \\ bS & -aS \\ (b/c)(dC-c) & -(ad/c)C \\ -(bd/c)S & -(ad/c)S \\ \\ b/c & -(a/c) \\ a/c & -(a/c) \\ b/c & -(b/c) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \mathbf{T}^T \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \mathbf{T}^T \mathbf{x}_2 \qquad (51)$$

Therefore, the remaining portion of the terminal equations involving the terminal forces and torques are

$$\mathbf{y}_2 = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = -\mathbf{T} \begin{bmatrix} f_{G1x} \\ f_{G1y} \\ f_{G2x} \\ f_{G2y} \\ f_{Gx} \\ f_{Gy} \\ \\ M \\ M_a \\ M_b \end{bmatrix} = -\mathbf{T}\,\mathbf{y}_1 \qquad (52)$$

On the other hand, the terminal equations of $\mathcal{R}_{123}$ given in equations (51) and (52) i.e.,

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} & \mathbf{T}^T \\ -\mathbf{T} & \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{x}_2 \end{bmatrix} \qquad (53)$$

when used in their present form, according to the procedure in selecting the proper tree discussed in section 2, the edges $(a_z, 0)$ and $(b_z, 0)$ be chords and all other nine edges be branches. With this form of the terminal equations for $\mathcal{R}_{123}$, the system graph can be drawn as in Figure 3f where the branches of the formulation tree are indicated by heavy lines. Although there are nine two-terminal mass and inertia components, and hence nine possible state variables, since only the edges corresponding to inertia components (J_1) and (J_2) can be taken as branches, there will be only two state variables ω_{J_1} and ω_{J_2} in the system graph.

Actually, as equation (51) states, one cannot increase the number of state variables beyond two. Of course, other than the variables ω_{J_1} and ω_{J_2}, a different pair of state variables can be selected by the rearrangement of equation (51).

The state equations of the system will be obtained by considering the terminal equations of two-terminal components (J_1) and (J_2) :

$$\begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \omega_{J_1} \\ \omega_{J_2} \end{bmatrix} = \begin{bmatrix} M_{J_1} \\ M_{J_2} \end{bmatrix} , \qquad (\, \omega_{J_1} = \omega_1 \, , \, \omega_{J_2} = \omega_2 \,) \qquad (54)$$

From the system graph in Figure 3f, the cut-set equations defined by the branches (J_1) and (J_2) together with the equation in (52) (where the variable f_{G1y} is omitted due to the corresponding zero column), we have

$$\begin{bmatrix} M_{J_1} \\ M_{J_2} \end{bmatrix} = -\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} =$$

$$\begin{bmatrix} -b & b(C-1) & bS & (b/a)(dC-c) & (bd/c)S & b/c & a/c & b/c \\ 0 & -aC & -aS & -(ad/c)C & -(ad/c)S & -(a/c) & -(a/c) & -(b/c) \end{bmatrix} \begin{bmatrix} f_{G1x} \\ f_{G2x} \\ f_{G2y} \\ f_{Gx} \\ f_{Gy} \\ M \\ M_a \\ M_b \end{bmatrix} \quad (55)$$

By using the terminal equations of two-terminal mass and inertia components and the terminal equations in (39), all the variables in the last column matrix of equation (55) can be expressed in terms of the velocity variables appearing in the left hand side of equation (51). Then, using the circuit equations, all these variables are expressed in terms of only the state variables ω_1 and ω_2 and the driving functions (gravitational forces $w_1 = -m_1 g, w_2 = -m_2 g, w = -mg$) as :

$$\left.\begin{array}{llll}
f_{G1x} & = -f_{m1x} & = & -m_1 \frac{d}{dt} v_{m1x} = -m_1 \frac{d}{dt} v_{G1x} = -m_1 \frac{d}{dt}(-b\omega_1) = bm_1 \frac{d}{dt}\omega_1 \\
f_{G2x} & = -f_{m2x} & = & -m_2 \frac{d}{dt} v_{m2x} = -m_2 \frac{d}{dt} v_{G2x} = -m_2 \frac{d}{dt} b(C-1)\omega_1 - aC\omega_2| \\
 & & = & -m_2 b(C-1)\frac{d}{dt}\omega_1 + m_2 ac\frac{d}{dt}\omega_2 + (m_2/c)S(b\omega_1 - a\omega_2)^2 \\
f_{G2y} & = -f_{m2y} + w_2 & = & -m_2 \frac{d}{dt} v_{m2y} + w_2 = -m_2 \frac{d}{dt}(bS\omega_1 - aS\omega_2) - m_2 g \\
 & & = & -m_2 bS\frac{d}{dt}\omega_1 + m_2 aS\frac{d}{dt}\omega_2 - (m_2/c)C(b\omega_1 - a\omega_2)^2) - m_2 g \\
f_{Gx} & = -f_m & = & -m\frac{d}{dt} v_{mx} = -m\frac{d}{dt} v_{Gx} = -m\frac{d}{dt}(b/c)(dC-c)\omega_1 - (ad/c)C\omega_2| \\
 & & = & -m(b/c)(dC-c)\frac{d}{dt}\omega_1 + m(ad/c)C\frac{d}{dt}\omega_2 + m(d/c^2)S(b\omega_1 - a\omega_2)^2 \\
f_{Gy} & = -f_{my} + w & = & -m\frac{d}{dt} v_{my} + w = -m\frac{d}{dt} v_{Gy} - mg = -m(d/c)\frac{d}{dt}(bS\omega_1 - aS\omega_2) \\
 & & = & -m(bd/c)S\frac{d}{dt} + m(ad/c)S\frac{d}{dt}\omega_2 - m(d/c^2)C(b\omega_1 - a\omega_2)^2 \\
M & = -M_J & = & -J\frac{d}{dt}\omega_J = -J\frac{d}{dt}\omega = -(J/c)\frac{d}{dt}(b\omega_1 - a\omega_2) \\
M_a & = -M_{B_a} & = & -B_a\omega_{B_a} = -B_a\omega_a = -B_a(a/c)(\omega_1 - \omega_2) \\
M_b & = -M_{B_b} & = & -B_b\omega_{B_b} = -B_b\omega_b = -B_b(b/c)(\omega_1 - \omega_2)
\end{array}\right\} \quad (56)$$

Substituting equations (56) into equation (55) and then into (54), after collecting terms corresponding to the variables ω_1, ω_2, $\frac{d}{dt}\omega_1$ and $\frac{d}{dt}\omega_2$, we obtain

$$\begin{bmatrix} M_{11}(\phi) & M_{12}(\phi) \\ M_{12}(\phi) & M_{22}(\phi) \end{bmatrix} \frac{d}{dt}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = B\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} N_1(\phi) & N_2(\phi) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}(b\omega_1 - a\omega_2) + \begin{bmatrix} L_1(\phi) \\ L_2(\phi) \end{bmatrix} \quad (57)$$

where

$$\left.\begin{array}{lcl} M_{11}(\phi) & = & J_1 + m_1 b^2 + 2m_2 b^2(1-C) + m(b/c)^2(d^2+c^2-2cdC) + (b/c)^2 J \\ M_{12}(\phi) & = & -m_2 ab(1-C) - m(abd/c^2)(d-cC) - (ab/c^2)J \\ M_{22}(\phi) & = & J_2 + m_2 a^2 + m(ad/c)^2 + (a/c)^2 J \\ B & = & (1/c)^2(a^2 B_a + b^2 B_b) \\ N_1(\phi) & = & -m_2 + (d/c)m|b^2 S \\ N_2(\phi) & = & \ \ m_2 + (d/c)m|abS \\ L_1(\phi) & = & -m_2 + (d/c)m|bgS \\ L_2(\phi) & = & \ \ m_2 + (d/c)m|agS \end{array}\right\} \quad (58)$$

In equation (57) most of the entries are the functions of only one variable, ϕ. Since $\omega = \frac{d}{dt}\phi$ does not appear as a state variable in this equation, either by using the relation in (48) all the entries are expressed in terms of the variables θ and ψ, both of which related to the state variables $\omega_1 = \frac{d}{dt}\theta$ and $\omega_2 = \frac{d}{dt}\psi$; or keeping ϕ as is, one of the present state variables, say ω_2, is changed to $\omega = \frac{d}{dt}\phi$. In the first case, the number of augmented state variables will be four $(\omega_1, \omega_2, \theta, \psi)$, while in the second case this number will be three (ω_1, ω, ϕ). To achieve this change of variables consider the relation in equation (48) which is expressed as

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (b/a) & -(c/a) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega \end{bmatrix} = \mathbf{P} \begin{bmatrix} \omega_1 \\ \omega \end{bmatrix} \quad (59)$$

where $\mathbf{P}$ is a constant transformation matrix. Substituting equation (59) into equation (57), and to keep the form of the system inertia matrix on the left hand side of equation (57) as symmetric, we premultiply both sides of this equation by $\mathbf{P}^T$, and obtain the desired state equations in which the angular velocities ω_1 and ω are now the state variables :

$$\begin{bmatrix} M'_1(\phi) & M'_{12}(\phi) \\ M'_{12}(\phi) & M'_{22}(\phi) \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \omega_1 \\ \omega \end{bmatrix} = (c/a)B \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega \end{bmatrix} + \begin{bmatrix} N'_1(\phi) & N'_2(\phi) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega \end{bmatrix} \omega + \begin{bmatrix} L'_1(\phi) \\ L'_2(\phi) \end{bmatrix} \quad (60)$$

where

$$\left.\begin{array}{lcl} M'_{11}(\phi) & = & J_1 + (b/a)^2 J_2 + b^2(m_1+m_2+m) \\ M'_{12}(\phi) & = & -(b/c)m_2 + (d/c)m|C - (bc/a^2)J_2 \\ M'_{22}(\phi) & = & J + (c/a)^2 J_2 + m_2 c^2 + md^2 \\ N'_1(\phi) & = & 0 \\ N'_2(\phi) & = & -bcm_2 + (d/c)m|S \\ L'_1(\phi) & = & 0 \\ L'_2(\phi) & = & -cm_2 + (d/c)m|gS \end{array}\right\} \quad (61)$$

In order to obtain the final form of the state equations, the relation $\frac{d}{dt}\phi = \omega$ must be added to equation (60).

A Special Case [25]: Assume that the component $\mathcal{R}$ in the mechanical system of Figure 3a does not exist ($m = 0$, $J = 0$, $B_a = B_b = 0$) and also $\mathcal{R}_1$ and $\mathcal{R}_2$ are cylindrical shells where $\mathcal{R}_2$ rolls inside of $\mathcal{R}_1$ without slipping ($J_1 = m_1 b^2$, $J_2 = m_2 a^2$) while $\mathcal{R}_1$ rolls on a horizontal plane with no skid. In this special case equation (60) takes of the form :

$$\begin{bmatrix} 2b^2(m_1+m_2) & -bcm_2(1+C) \\ -bcm_2(1+C) & 2c^2 m_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \omega_1 \\ \omega \end{bmatrix} = \begin{bmatrix} -bcm_2 S\omega^2 \\ -cgm_2 S \end{bmatrix} \quad (62)$$

On the other hand, in this special case, the mechanical system in Figure 3a becomes conservative and the state variables can be decomposed. Indeed, the first row in equation (62) is integrable twice yielding

$$2b^2(m_1+m_2)\,\omega_1 - bcm_2(1+C)\,\omega \;=\; k_1 \tag{63}$$

and

$$2b^2(m_1+m_2)\,\theta - bcm_2(\phi+\sin\phi) \;=\; k_1 t + k_2 \tag{64}$$

where k_1 and k_2 are integration constants to be determined from the initial conditions on $\theta(0)$, $\phi(0)$, $\omega_1(0)$ and $\omega(0)$. If $\omega_1(0)=\omega(0)=0$, or if $\omega_1(0)=0$, $\phi=180°$, then k_1 vanishes and from equations (63) and (64), the variables θ and $\omega_1=\frac{d}{dt}\theta$ are expressible in terms of the variables ϕ and $\omega=\frac{d}{dt}\phi$ and the constant k_2. Therefore, from the second row of equation (62), with Pars' notation, we have

$$\left[1-\beta^2(1+C)^2\right]\frac{d}{dt}\omega \;=\; -(1/2)n^2S-\beta^2(1+C)S\,\omega^2 \tag{65}$$

where

$$\beta^2 \;=\; \frac{m_2}{4(m_1+m_2)} \quad , \quad n^2 \;=\; \frac{g}{c}$$

Hence, equation (65) together with the relation $\frac{d}{dt}\phi=\omega$ gives the decomposed equation of motion in the following form :

$$\frac{d}{dt}\begin{bmatrix}\omega\\ \phi\end{bmatrix} = \begin{bmatrix}F(\omega,\,\phi)\\ \omega\end{bmatrix} \tag{66}$$

with

$$F(\omega,\,\phi) \;=\; -\frac{1}{1-\beta^2(1+C)^2}\left[(1/2)n^2S-\beta^2(1+C)S\,\omega^2\right] \tag{67}$$

Example - 3 Two fixed points, A and B, of a vertical thin lamina shown in Figure 4a, are constraint to slide along the fixed perpendicular (Oy) and (Ox) axes, respectively. Two linear spring components are also attached to the lamina at these points. The viscous frictions (B_1) and (B_2) along the axes will also be considered. Obtain the state equations of this system.

Considering the lamina as an ideal multiport component, the schematic diagram of the system can be drawn as in Figure 4b. The terminal equations of the ideal lamina is of the form :

$$\begin{bmatrix}\mathbf{x}_A\\ \mathbf{x}_B\\ \\ \mathbf{y}_G\end{bmatrix} = \begin{bmatrix} & & \mathbf{K}_A^T\\ & 0 & \mathbf{K}_B^T\\ & & \\ -\mathbf{K}_A & -\mathbf{K}_B & \end{bmatrix}\begin{bmatrix}\mathbf{y}_A\\ \mathbf{y}_B\\ \\ \mathbf{x}_G\end{bmatrix} \quad \text{where} \quad \mathbf{K}_i = \begin{bmatrix}1 & & \\ & 1 & \\ -r_{Giy} & r_{Gix} & 1\end{bmatrix} \quad (i=A,\,B) \tag{68}$$

From the terminal conditions shown in Figure 4b, a simplified five-port scalar model for the ideal lamina with terminals $(O, A_y, B_x, G_x, G_y, G_z)$ corresponding to the terminal graph in Figure 4c can be drived as

$$\mathbf{x} = \begin{bmatrix}v_{Ay}\\ v_{Bx}\\ v_{Gx}\\ v_{Gy}\end{bmatrix} = \begin{bmatrix}r_{ABx}\\ r_{ABy}\\ -r_{GAy}\\ r_{GBx}\end{bmatrix}\omega_{Gz} \quad , \quad M_{Gz} = \left[-\,r_{ABx} \;\; r_{ABy} \;\; -r_{GAy} \;\; r_{GBx}\right]\begin{bmatrix}f_{Ay}\\ f_{Bx}\\ f_{Gx}\\ f_{Gy}\end{bmatrix} \tag{69}$$

where $r_{ABx} = r_{GBx} - r_{GAx}$ and $r_{ABy} = r_{GBy} - r_{GAy}$. The relations between the terminal velocities appearing in equation (69) imply that only one edge of the terminal graph of the simplified model will be a chord, all other four edges will be branches. In order to obtain a maximum number of state variables, the form of the terminal equations in (69) is proper. We may also bring any one of the variables v_{Gx}, v_{Gy} from the left to the right hand side in equation (69), still the maximum of three state variables are obtained. However, if one of the variables v_{Ay} or v_{Bx} is moved to the right hand side of the first equation in (69), two state variables will be lost and only one state equation having much complicated expression will be obtained. Therefore, we shall obtain the state equations by considering the terminal equations of the inertia and two spring components :

$$\begin{bmatrix} J & & \\ & 1/K_1 & \\ & & 1/K_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \omega_J \\ f_{K_1} \\ f_{K_2} \end{bmatrix} = \begin{bmatrix} M_J \\ v_{K_1} \\ v_{K_2} \end{bmatrix} \tag{70}$$

where the variables M_J, v_{K_1} and v_{K_2} must be expressed in terms of the state variables ω_J, f_{K_1}, f_{K_2} and the driving function $w = -mg$. Note that, from the system graph in Figure 4d and the first equation in (69)

$$\left.\begin{array}{lllll} v_{K_1} & = & v_{GAy} & = & (r_{GBx} - r_{GAx})\omega_J \\ v_{K_2} & = & v_{GBx} & = & (r_{GBy} - r_{GAy})\omega_J \end{array}\right\} \tag{71}$$

On the other hand, for M_J in equation (70), again from the system graph and equation (69) we have $M_J = -M_{Gz}$ where the variables f_{Ay}, f_{Bx}, f_{Gx} and f_{Gy} are expressed as

$$\left.\begin{array}{lll} f_{Ax} & = & -f_{K_1} - f_{B_1} = f_{K_1} - B_1 v_{Ay} = -f_{K_1} - B_1(r_{GBx} - r_{GAx})\omega_J \\ f_{By} & = & -f_{K_2} - f_{B_2} = -f_{K_2} - B_2 v_{B_2} = -f_{K_2} - B_2 v_{By} = -f_{K_2} - B_2(r_{GBy} - r_{GAy})\omega_J \\ f_{Gx} & = & -f_{mx} = -m\frac{d}{dt}v_{mx} = -m\frac{d}{dt}v_{Gx} = m\frac{d}{dt}(r_{GAy}\omega_J) \\ f_{Gy} & = & -f_{my} + w = -m\frac{d}{dt}v_{my} + w = -m\frac{d}{dt}v_{Gy} - mg = -m\frac{d}{dt}(r_{GBx}\omega_J) - mg \end{array}\right\} \tag{72}$$

therefore for the expression of M_J, we have

$$M_J = -r_{ABx}f_{K_1} - r_{ABy}f_{K_2} - (B_1 r_{ABx}^2 + B_2 r_{ABy}^2) - m(r_{Ay}^2 + r_{Bx}^2)\frac{d}{dt}\omega_J - (1/2)m\omega_J\frac{d}{dt}(r_{Ay}^2 + r_{Bx}^2) - r_{Bx}mg \tag{73}$$

Substituting equations (71) and (73) into equation (70), the state equations of the system will be obtained in the following form :

$$\begin{bmatrix} M(\theta) & & \\ & 1/K_1 & \\ & & 1/K_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \omega_J \\ f_{K_1} \\ f_{K_2} \end{bmatrix} = \begin{bmatrix} B(\theta) & -r_{ABx} & -r_{ABy} \\ r_{ABx} & 0 & 0 \\ r_{ABy} & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_J \\ f_{K_1} \\ f_{K_2} \end{bmatrix} + \begin{bmatrix} N(\theta)\omega_J^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} L(\theta) \\ 0 \\ 0 \end{bmatrix} \tag{74}$$

Since from Figure 4a we have

$$\left.\begin{array}{lll} r_{GAy} & = & a\sin(\theta - \alpha) \\ r_{GBx} & = & b\cos(\theta + \beta) \\ r_{ABx} & = & \ell\cos\theta \\ r_{ABy} & = & -\ell\sin\theta \end{array}\right\} \tag{75}$$

then the various coefficients in equation (74) have the following explicit expressions :

$$\left.\begin{array}{lll} M(\theta) & = & J + m(r_{GAy}^2 + r_{GBx}^2) = J + m\left[a^2\sin^2(\theta - \alpha) + b^2\cos^2(\theta - \alpha)\right] \\ N(\theta) & = & m\left[a^2\sin(\theta - \alpha)\cos(\theta - \alpha) - b^2\sin(\theta + \beta)\cos(\theta + \beta)\right] \\ B(\theta) & = & -\ell^2(B_1\cos^2\theta + B_2\sin^2\theta) \\ L(\theta) & = & -mbg\cos(\theta + \beta) \end{array}\right\} \tag{76}$$

Equation (74) must be augmented to include the relation $\frac{d}{dt}\theta = \omega_{B_2} = \omega_{G2} = \omega_J$ yielding the final set of fourth order state equations in the **variables** $\boldsymbol{\theta}, \omega_J, f_{K_1}$ and f_{K_2}.

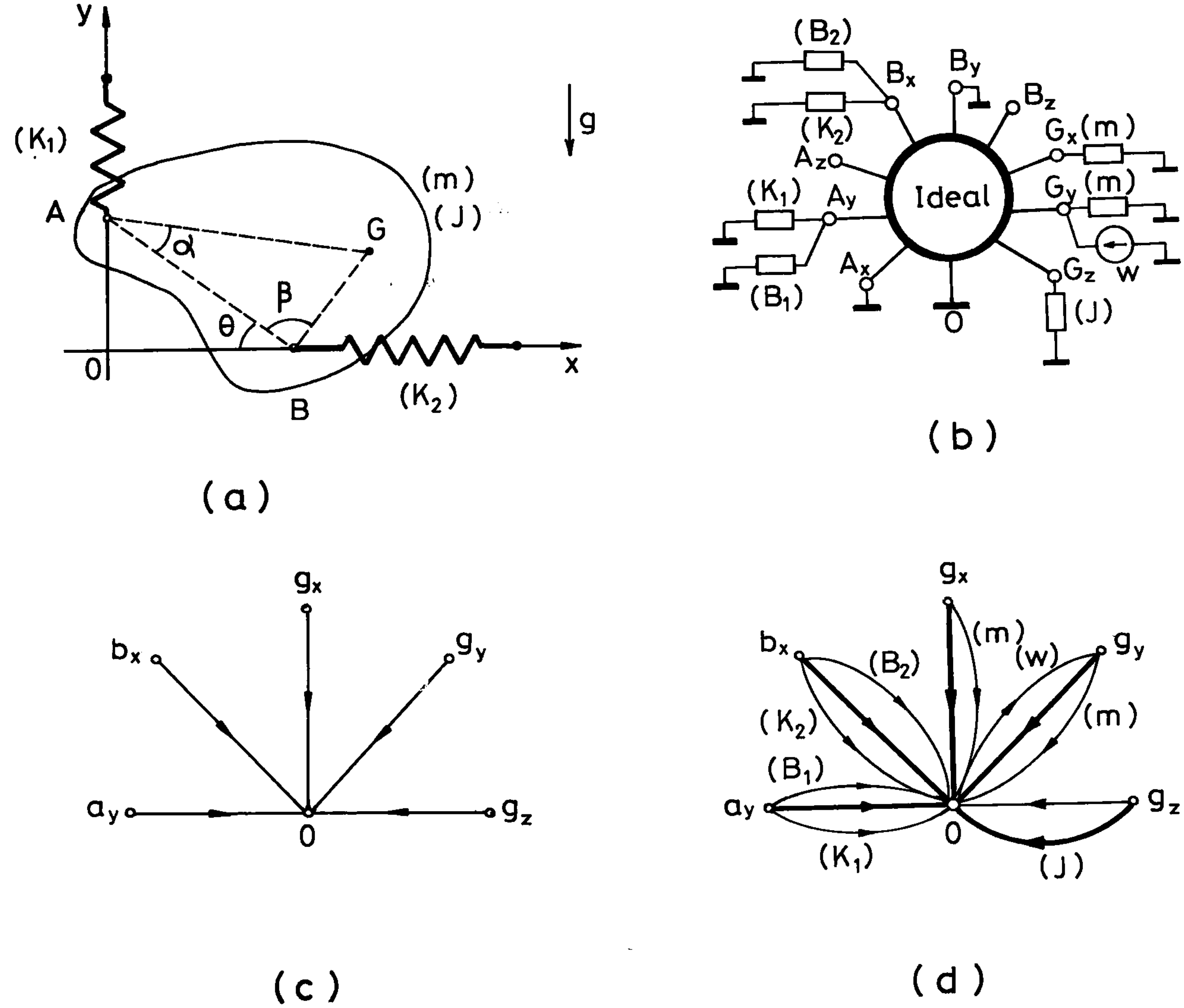

Figure - 4 (a) **A mechanical system in two-dimensional motion containing a lamina, two springs and two viscous** dampers. (b) Schematic diagram of the mechanical system in (*a*) where the lamina is idealized and the terminal conditions are indicated. (c) The terminal graph of the simplified ideal component (lamina). (d) The system graph with a proper tree (indicated by heavy lines).

5. Conclusions and Discussion

The method used in deriving the state equations for the mechanical systems in planar motion, which is applied here to three different systems in detail, utilizes the system graph and a proper formulation tree selected in it as an important tool. It furnishes a complete set of information about the interconnection pattern of the components forming the system. In order to ease the selection of the state variables, based on a proper tree, each multiport component is represented by the interconnected components; one multiport ideal component and a number of two-terminal components. In planar motion, the ideal components correspond to ideal transformers in electrical networks. On the other hand, interconnection pattern of the system can also be interpreted as an ideal transformer. Indeed, e.g., an electronic circuit is usually built by soldering the terminals of various electrical components within the holes of a PC (printed circuit) board and the PC board itself is viewed as a multiport *connection box* containing nothing but several short-circuits and open-circuits in it [14]. The terminal equation of a connection box is exactly in the form of equation (9) where $\mathbf{K}^T$ is the submatrix of a circuit matrix while $-\mathbf{K}$ is the submatrix of the corresponding cut-set matrix. This representation goes back early to Kron's time [26]. For mechanical systems in three-dimensional motion, a formulation procedure, not leading to state equations, is given in [3] where the systems contain interconnected rigid bodies only. Publications utilizing essentially the same approach for the mechanical systems containing also the components other than the rigid bodies have already appeared in [11],[27],[28],[29]. The present technique have been used since the early sixties [5], [6],[7],[8] for the electrical networks and other type of lumped physical systems including the mechanical systems in one dimensional motion. However, the extension of this approach to three-dimensional mechanical systems evolved rather slowly. By this time, in the seventies, development of an equivalent technique, called the *Bond Graphs*, which is well known more to mechanical engineers, took place [30]. Bond graphs carries the same information as the system graph of a mechanical system and utilizes essentially *modulated transformers and gyrators: MTF, MGY* which correspond to ideal components (perfect couplers) in linear graph technique. Further, an important concept, namely the *causality* in bond graph approach correspond to the selection of a formulation tree in the system graph. This selection actually determines as to which variables are retained and which variables are eliminated in the final set of equations for that system.

References

[1] H.Goldstein, *Classical Mechanics.* Addison-Wesley Press, Inc., Cambridge. Mass., 1950.

[2] Y.Tokad, "A Network Model for Rigid Body Motion", *Dynamics and Control,* vol.2, No.1, pp.59-82, 1992.

[3] Y.Tokad, "Network Model Approach to the Analysis of Multirigid-Body Systems", *Dynamics and Control,* vol.3, No.2, pp.107-125, 1993.

[4] H.M.Trent, "Isomorphisms Between Oriented Linear Graphs and Lumped Physical Systems", *J.Acoust. Soc. Amer.,* vol.27, No.3, pp.500-527, 1955.

[5] H.E.Koenig and M.B.Reed, "Linear-graph representation of Multi-terminal elements", in *Proc. Natl. Electronics Conf.,* pp.661-674, Chicago, 1958.

[6] H.E.Koenig, Y.Tokad and H.K.Kesavan, *Analysis of Discrete Physical Systems,* McGraw-Hill: New York, 1967.

[7] P.H.O'N.Roe, *Networks and Systems,* Addison Wesley Publishing Co., Inc., Reading, Mass., 1966.

[8] H.E.Koenig and W.A.Blackwell, *Electromechanical System Theory,* McGraw-Hill: New York, 1961.

[9] K.Abdullah and Y.Tokad, "On the Existence of Mathematical Models for Multiterminal *RC*Γ Networks", *IEEE Trans. Circuit Theory,* vol.CT-19, No,5, pp.419-424, 1972.

[10] S.Seshu and M.B.Reed, *Linear Graphs and Electrical Networks,* Addison Wesley Publishing Co., Inc., Reading, Mass., 1961.

[11] J.C.K.Chou, H.K.Kesavan and K.Singhal, "A System Approach to Three-Dimensional Multibody Systems Using Graph-Theoretic Models", *IEEE Trans. Syst., Man, Cyber.*, vol.SMC-12, No.2, pp.219-230, March/April 1986.

[12] Y.Tokad and T.Zeren, "On the Terminal Solvability of *RLCT* Networks", *METU Journal of Pure and Applied Sciences*, vol.4, No.3, pp.349-374, December 1971.

[13] V.Belevitch, "On the Algebraic Structure of Formal Realisability Theory", *Revue Haut Fréquence*, vol.4, No.8, pp.183-194, 1959.

[14] V.Belevitch, *Classical Network Theory*, Holden-Day, San Fransisco, 1968.

[15] R.W.Newcomb, *Linear Multiport Synthesis*, McGraw-Hill Book Co., New York, 1966.

[16] Y.Tokad, *Foundations of Passive Electrical Network Synthesis*, Publication No.41, Faculty of Engineering, Middle East Technical University, vol.2, 1972.

[17] H.M.Trent, "On the Connection Between the Properties of Oriented Linear Graphs and Analysis of Lumped Physical Systems", *Journal of Research of the National Bureaux of Standards*, vol.698, Nos.1 and 2, pp.79-84, 1965.

[18] M.Milic, "General Passive Networks-Solvability, Degeneracies and Order of Complexity", *IEEE Trans. Circuits and Systems*, vol.CAS-21, No.2, pp.117-123, 1974.

[19] Y. Tokad, "On the Topological Conditions for Linear Lumped Time Invariant Networks Containing Multi-terminal Components", *Bulletin of the Technical University of Istanbul*, vol.40, No.2, pp.479-496, 1987.

[20] Y.Tokad, "State Variable Technique for the Analysis and Synthesis of Electrical Networks", in *Network and System Theory*, PLL Conference Publication 12 (Eds. J.K.Skwirzynski and J.O.Scanlan), pp.83-88, NATO Advanced Study Institute, Bournemouth, U.K., September 1972.

[21] J.Helszjn, *Nonreciprocal Microwave Junctions and Circulators*, John Wiley and Sons, New York, 1975.

[22] T.R.Bashkow, "The *A* Matrix, New Network Description" *IRE Transactions on Circuit Theory*, vol.CT-4, No.3, pp.117-119, September 1957.

[23] P.R.Bryant, "The Order of Complexity of Electrical Networks", *Proc. IEE (London)*,vol.106 C, Monograph 335 E, pp.174-178, June 1959.

[24] W.D.MacMillan, *Theoretical Mechanics*, Dover Publications, Inc., Newyork, 1960.

[25] L.A.Pars, *Analytical Dynamics*, Heinemann, London, 1965.

[26] G.Kron, *Tensor Analysis of Networks*, John Wiley and Sons, Inc., New York, 1939.

[27] G.C.Andrews and H.K.Kesavan, "The Vector-Network Model: A New Approach to Vector Dynamics", *Mechanism and Machine Theory*, vol.10, pp.57-75, 1975.

[28] K.Singhal and H.K.Kesavan, "Dynamic Analysis of Mechanism Via Vector Network Model", *Mechanism and Machine Theory*, vol.18, No.3, pp.175-180, 1983.

[29] K.Singhal, H.K.Kesavan and Z.I.Ahmad, "Vector-Network Model for Kinematics: The 4-Bar Mechanism", *Mechanism and Machine Theory*, vol.18, No.5, pp.363-369, 1983.

[30] D.Karnopp and R.Rosenberg, *System Dynamics: A Unified Approach*, John Wiley and Sons, New York, 1975.

Appendix

Realization of 3-port Component ($\mathcal{S}$) *with the Terminal Equations in (7):*

Consider the expression of the *moment of momentum* (angular momentum) $\mathbf{h} = \mathbf{J}\boldsymbol{\omega}$, where the inertia matrix $\mathbf{J}$ is diagonal. Therefore, equation (7) can be written as

$$\mathbf{M} = \boldsymbol{\Omega}\mathbf{J}\boldsymbol{\omega} = \boldsymbol{\Omega}\mathbf{h} = \mathbf{H}^T\boldsymbol{\omega} \tag{77}$$

where

$$\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \mathbf{J}\boldsymbol{\omega} = \begin{bmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_3 \end{bmatrix}$$

$$\mathbf{H}^T = -\mathbf{H} = \begin{bmatrix} 0 & h_3 & -h_2 \\ -h_3 & 0 & h_1 \\ h_2 & -h_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I_{33}\omega_3 & -I_{22}\omega_2 \\ -I_{33}\omega_3 & 0 & I_{11}\omega_1 \\ I_{22}\omega_2 & -I_{11}\omega_{11} & 0 \end{bmatrix}$$

$\mathbf{H}^T$ can be decomposed as the sum of three simpler skew symmetric matrices

$$\mathbf{H}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & h_1 \\ 0 & -h_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -h_2 \\ 0 & 0 & 0 \\ h_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & h_3 & 0 \\ -h_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{78}$$

in which each (admittance) matrix corresponds to a 2-port gyrator, i.e., realization of 3-port component with the terminal equation in (7) is as shown in Figure 1f. Note that the gyrator parameters (gyration constants) h_1, h_2, h_3 are actually not constant but proportional to the components of the angular velocity $\boldsymbol{\omega}$. For example, the terminal equation of the gyrator corresponding to the first term in equation (78) is

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & h_1 \\ 0 & -h_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 0 \\ h_1\omega_3 \\ -h_1\omega_2 \end{bmatrix} = I_{11} \begin{bmatrix} 0 \\ \omega_1\omega_3 \\ -\omega_1\omega_2 \end{bmatrix}$$

which represents a nonlinear, nonenergic (ideal) 3-port component.

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Mekanik Sistemlerin Yüzeysel Hareket Denklemlerinin Gösterimi

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Özet

Bu makalede mekanik sistemlerin hareket denklemlerinin elektrik mühendisliğinde kullanılan yerleşik yöntemler kullanılarak durum denklemleri olarak ifade edilme olanakları sunulmaktadır.

Kullanılan yaklaşımın kısa bir özeti verildikten sonra üç ayri örnek ayrıntılı olarak irdelenmektedir. Konuyu karmaşıklaştırmamak amacı ile örneklerde kullanılan mekanik sistemlerin yatay yüzey üzerindeki hareketleri ele alınmıştır. Ancak aynı teknikler üç boyutta hareket eden mekanik sistemler için de geçerlidir.