# Existence of Chaos in the Fractional Order Unified System: The Lowest Effective Dimension

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## Abstract

**This paper deals with the existence of chaos in the fractional order unified system. In an earlier published work, based on the numerical simulation results it is claimed that the lowest effective dimension in the fractional order unified system in order to present chaotic behaviors is 2.76. In this paper, it is analytically shown that the fractional order unified system can be chaotic for effective dimensions lower than the claimed lowest effective dimension. Also, the lowest effective dimension for existence of chaos in such a system is analytically determined. Computer simulation results, obtained based on a reliable numerical method, are brought to back up the presented analytical work in the paper.**

## 1. Introduction

In recent years, investigating chaotic behaviors in the fractional order systems has attracted increasing attention in research studies. In many of published works in this subject, existence of chaos in a special fractional order system has been reported. The fractional order Chua circuit [1], the fractional order Duffing system [2], the fractional order Lorenz system [3], the fractional order Rössler system [4], the fractional order Chen system [5], the fractional order Ikeda delay system [6] and non-integer order cellular neural networks [7] are some of chaotic fractional order systems introduced in literature. In the most of the mentioned papers, a numerical simulation based approach has been presented to prove existence of chaos. Since numerical methods are not always reliable for detecting chaos [8, 9], use of analytical based approaches are requires as well to support the claim of existing of chaos in a fractional order system. Taking this point in mind, in this paper an approach combining mathematical analysis and numerical calculations is presented to investigate the existence of chaos in the fractional order unified system. Furthermore, the lowest effective dimension for existence chaos in such a system is determined, and it is verified that the calculated dimension is lower than the one previously reported in literature.

The paper is organized as follows. Section 2 presents some remarks on the unified chaotic system introduced by Lü and his colleagues in [10]. In Section 3, a necessary condition for existence of chaos in the fractional order counterpart of the unified system is derived. This condition is used to obtain the lowest effective dimension in which the fractional order unified system could exhibit chaotic behaviors. The related numerical simulation results are brought in Section 4. Finally, the paper is concluded in Section 5.

## 2. Some Remarks on the Unified System

The unified system, introduced by Lü and his colleagues in [10] represents the continued transition from the Lorenz system to the Chen system. This system is described by

$$\begin{cases} \frac{dx}{dt} = (25\delta + 10)(y - x) \\ \frac{dy}{dt} = (28 - 35\delta)x - xz + (29\delta - 1)y \\ \frac{dz}{dt} = xy - \frac{\delta + 8}{3}z \end{cases}, \quad (1)$$

where $\delta$ is an arbitrary value in range $[0, 1]$. System (1) has three fixed points as follows.

$$\begin{aligned} &O = (0, 0, 0), \\ &S^{\pm} = (\pm\sqrt{(8+\delta)(9-2\delta)}, \pm\sqrt{(8+\delta)(9-2\delta)}, 27-6\delta) \end{aligned}. \quad (2)$$

One can easily show that the linearized model of system (1) around the fixed point $O$ has the following characteristic polynomial [10].

$$(s + \frac{41 - 11\delta}{3})(s^2 + (11 - 4\delta)s + (25\delta + 10)(6\delta - 27)) = 0 . \quad (3)$$

Since $0 \le \delta \le 1$, we have $-(41 - 11\delta)/3 < 0$ and $(25\delta + 10)(6\delta - 27) < 0$. Hence, the linearized model of system (1) around the fixed point $O$ has two negative and one positive eigenvalues. This means that the fixed point $O$ is a saddle point of index 1 for the unified system. In a 3D nonlinear dynamical system, a saddle point is an equilibrium point on which the equivalent linearized model has two eigenvalues with negative real parts and one eigenvalue with positive real part. In the same system, an equilibrium point is called saddle point of index 2 if one of its corresponding eigenvalues has negative real part and the real parts of other eigenvalues are positive.

The characteristic polynomial for the linearized model of system (1) around fixed points $S^{\pm}$ is given as follows [10].

$$\begin{aligned} s^3 + \frac{41 - 11\delta}{3}s^2 + \frac{(38 - 10\delta)(\delta + 8)}{3}s + \\ 2(25\delta + 10)(\delta + 8)(9 - 2\delta) = 0 \end{aligned}. \quad (4)$$

Fig. 1 shows the root locus of characteristic polynomial (4) when $\delta$ continuously changes form 0 to 1. According to this figure, it concluded that the fixed points $S^{\pm}$ are saddle points of index 2 for system (1). In 3D chaotic systems, it is shown that scrolls are generated only around the saddle points of index 2. Moreover, saddle points of index 1 are responsible for connecting scrolls [11-13]. Therefore, the two scrolls available in the chaotic attractor of unified system surround the fixed points $S^{\pm}$ and the fixed point $O$ is responsible to connect these scrolls.

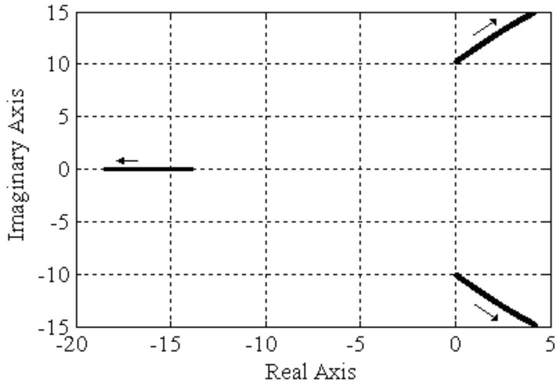



Fig. 1: Root locus of characteristic polynomial (4) when $\delta$ continuously changes form 0 to 1.

## 3. A Necessary Condition for Existence of Chaos in the Fractional Order Unified System

As mentioned before, finding chaotic behaviors by numerical simulations in different fractional order systems is subject of many papers published in the recent years. For instance, existence of chaos in fractional order unified system has been shown in [14]. The fractional order unified system is described by

$$\begin{cases} \dfrac{d^{\alpha}x}{dt^{\alpha}} = (25\delta + 10)(y - x) \\ \dfrac{d^{\alpha}y}{dt^{\alpha}} = (28 - 35\delta)x - xz + (29\delta - 1)y \\ \dfrac{d^{\alpha}z}{dt^{\alpha}} = xy - \dfrac{\delta + 8}{3}z \end{cases}, \qquad (5)$$

where $0 < \alpha \le 1$ is the fractional order of involved derivatives. The effective dimension (sum of the orders of all involved derivatives) of system (5) is equal to $3\alpha$. The fixed points of this system are same as those of the integer-order unified system. Suppose that $a_\delta \pm jb_\delta$ ( $a_\delta, b_\delta > 0$ ) are the corresponding unstable eigenvalues of fixed points $S^{\pm}$ of system (1) for a given $\delta$. It can be shown that a necessary condition for fractional order system (5) to remain chaotic is remaining the eigenvalues $a_\delta \pm jb_\delta$ in the unstable region, otherwise these fixed points becomes asymptotically stable and then attracts the nearby trajectories (For more details see [15, 16]). This means that system (5) can be chaotic only when the following condition is satisfied.

$$\tan(\alpha\frac{\pi}{2}) \ge \frac{b_\delta}{a_\delta} \quad \Rightarrow \quad \alpha \ge \frac{2}{\pi}\tan^{-1}(\frac{b_\delta}{a_\delta}) \qquad (6)$$

Therefore, for a given $\delta$, system (5) has the necessary condition to exhibit chaotic behavior if its effective dimension be more than $6\tan^{-1}(b_\delta / a_\delta)/\pi$. Fig. 2 demonstrates the region in which system (5) can be chaotic.

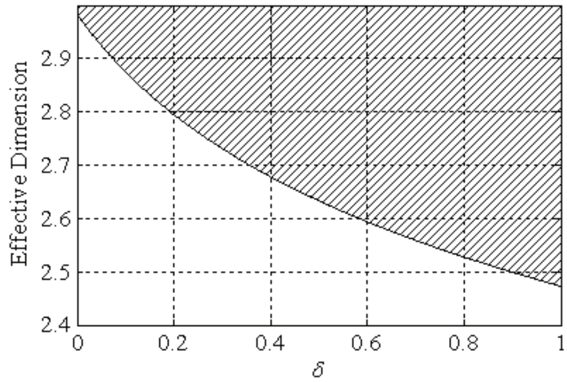
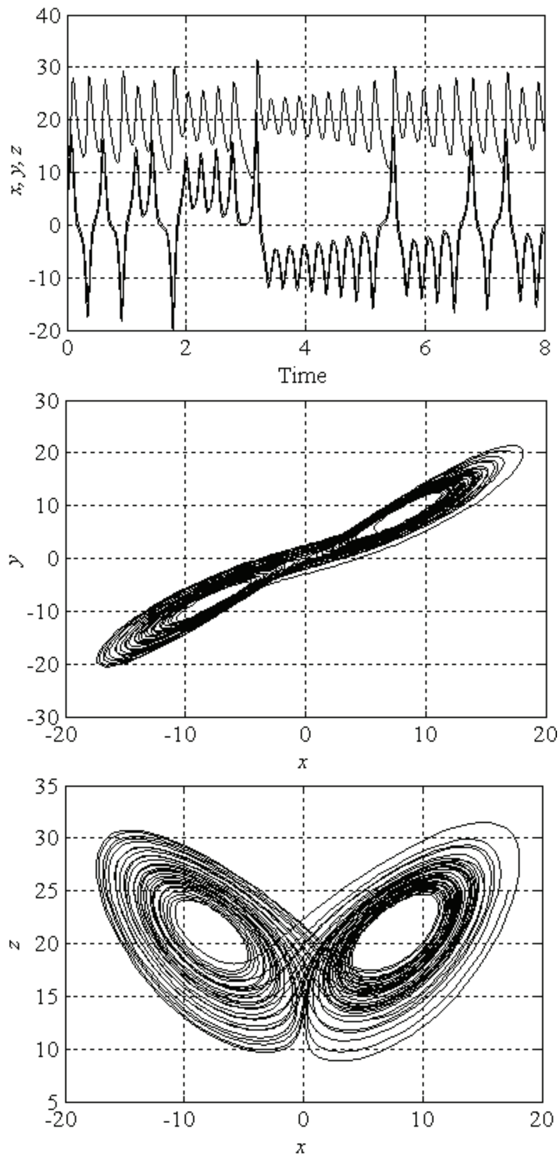



Fig. 2: The shadow region shows the region in which the system (5) may exhibit chaos.



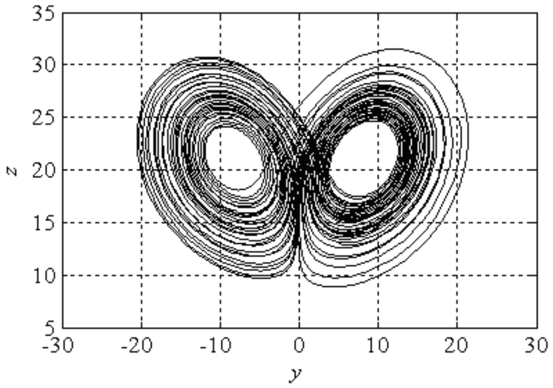



Fig. 3: Simulation results for fractional order unified system when $\delta=1$ and $\alpha=0.825$ (The effective dimension is 2.475).

## 4. Numerical Simulations

According to Fig. 2, the maximum effective dimension in which the system (5) may be chaotic has a direct relation with system parameter $\delta$. For the maximum value of parameter $\delta$, i.e. $\delta=1$, the fractional order $\alpha$ for which the fixed points $S^{\pm}$ become stable is calculated from (6) as 0.8244. Fig. 3 shows numerical simulation results for fractional order unified system when $\delta=1$ and $\alpha=0.825$. This figure confirms existence of chaos in fractional order unified system for the lowest effective dimension $3\times 0.825=2.475$. The largest Lyapunov exponent for this case is estimated as 0.968. Also, Fig. 4 shows numerical simulation results for fractional order unified system when $\delta=1$ and $\alpha=0.81$. In this case, the system trajectories converge to stable fixed points and consequently the system is not chaotic. The simulations of this section have been performed using the method introduced in [17] to find the solution of a Caputo definition based fractional differential equation. This method is an improved version of Adams-Bashforth-Moulton algorithm and is proved to be reliable for simulating the chaotic fractional order systems [8].

For other values of system parameter $\delta$, the maximum effective dimension for which the system (5) may exhibit chaotic behavior can be determined using equation (6). For instance, if $\delta=0.8$, the maximum fractional order for which the chaotic behavior exists is equal to 0.843, or equivalently the maximum effective dimension is equal to $3\times 0.843=2.529$. Fig. 5, which gives numerical simulation results for fractional order unified system when $\delta=0.8$ and $\alpha=0.843$, confirms the existence of chaos in the fractional-order unified system with effective dimension 2.529. In this case, the largest Lyapunov exponent is estimated as 0.553.

## 5. Conclusions

In this paper, we analytically found a lowest limit for the effective dimension in which the fractional-order unified system in parametric range $0\le\delta\le 1$ can be chaotic. This lowest limit is about 2.475. Numerical simulations confirmed the existence of chaos in the fractional-order unified system with this effective dimension. This order is less than the order found in paper [14], i.e. 2.76. Therefore, it seems the lowest order found in [14] should be modified.

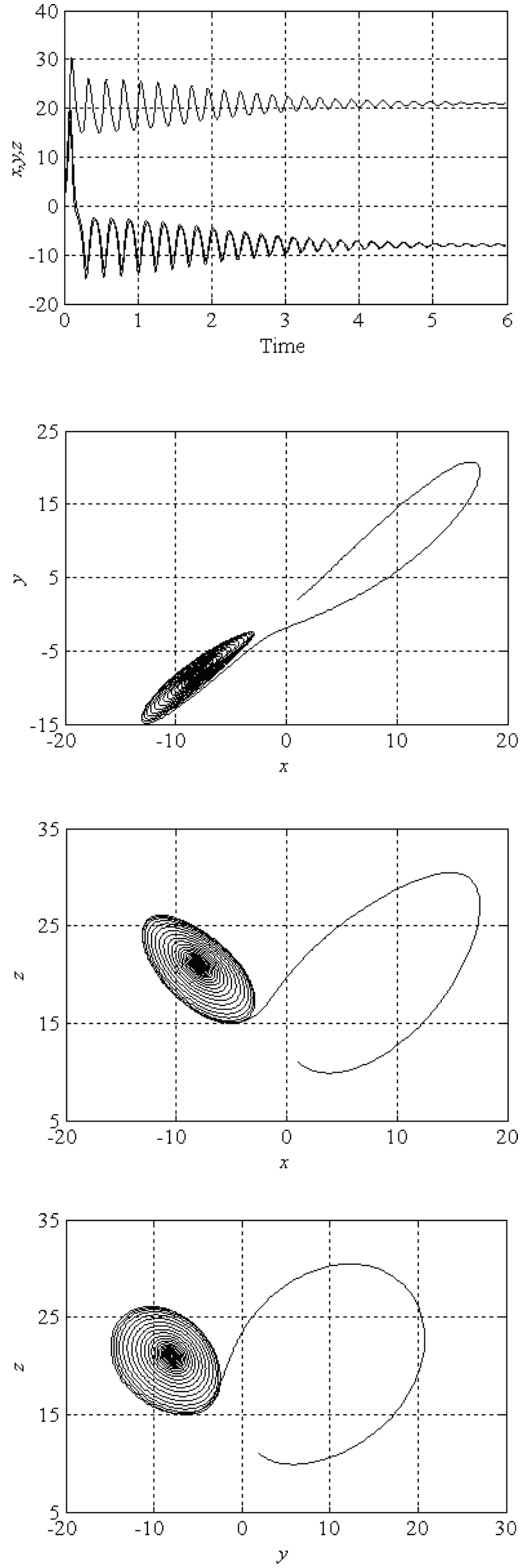



Fig. 4: Simulation results for fractional order unified system when $\delta=1$ and $\alpha=0.81$ (The effective dimension is 2.43).

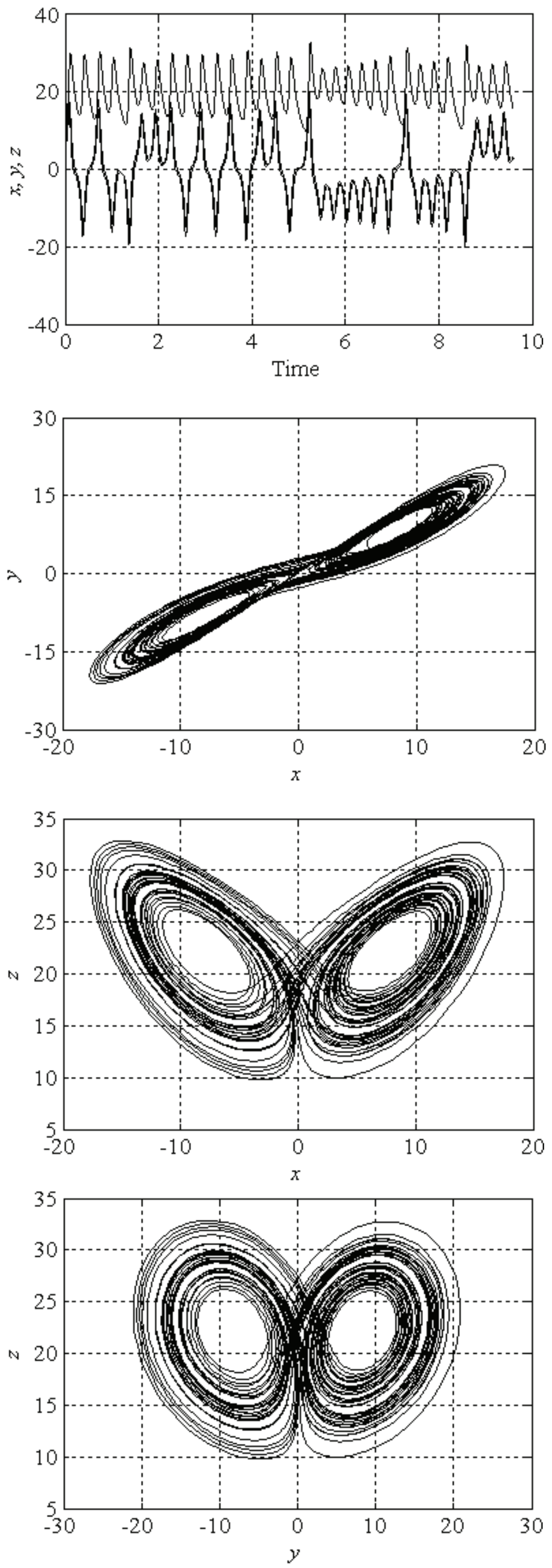



Fig. 5: Simulation results for fractional order unified system when $\delta = 0.8$ and $\alpha = 0.843$ (The effective dimension is 2.529).

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