Electromagnetic Scattering by Inhomogeneous Dielectric Bodies Modeled by Cubic Elements

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Abstract

The electric field distribution in an axially symmetric inhomogeneous dielectric body illuminated by a homogeneous plane wave is investigated by using a generalized analytical technique. The problem is formulated through the dyadic Green's function technique and solved by using the Method of Moments (MoM), which models an arbitrarily shaped dielectric body in terms of cubic elements. The singularity appearing in Green's functions is eliminated through a procedure which shapes the elements of MoM matrix into surface integrals that can be computed effectively. The numerical solutions are compared to analytical reference solutions for the special cases of homogeneous and concentric homogeneous spheres and the required size of the MoM matrix is observed to be quite small providing a rapid convergence. The convergence and the range of validity of the general analytical technique under consideration is tested for the first time for certain canonical problems of practical interest.

1. Introduction

Dyadic Green's functions are indispensable in the investigation of scattering by 3-D dielectric bodies of arbitrary shape (cf. [1]). The Green's function technique describes the total scattered field as a volume integral over the dielectric body. The fundamental difficulty with this technique is the singularity of the volume integral that occurs when the observation point falls into the region occupied by the scatterer. Many different procedures are available in literature for handling such singularities. For instance, [2] extracts a small region that includes the point of singularity and provides an analytical result for the singular term by integration in the Cauchy sense. [3-4] represents the electric field inside and outside the source region by generalized electric dyadic Green's functions which constitute a conventional dyadic determined outside and a source dyadic determined inside the region of singularity. [5] describes the derivatives of Green's functions in the distributional sense, whereas [6] uses these results to define a new formula which expresses the second derivative of Green's functions in the source region. While it is impractical to provide a satisfactory list of all such papers in the area, most relevant to our present investigation is a general analytical technique introduced by [7] in which the volume integrations are converted to surface integrations through an integral theorem, thereby eliminating the pole singularity.

Numerical solutions for the problem of scattering by dielectric objects were initiated by [8]. The Method of Moments (MoM) [9], which transforms integrodifferential equations to systems of linear equations, has been incorporated as a standard method in investigating such problems. Alternative geometrical models have been tested in modeling an arbitrary dielectric body with MoM. For instance, [10] uses MoM with cubic elements, whereas [11] uses tetrahedral elements and [12] uses hexahedral elements for modeling the scattering objects.

In this study, the procedure given in [7] for eliminating the singularity of dyadic Green's function is adopted for the problem of plane wave scattering by an arbitrary inhomogeneous dielectric body as depicted in Fig. 1.

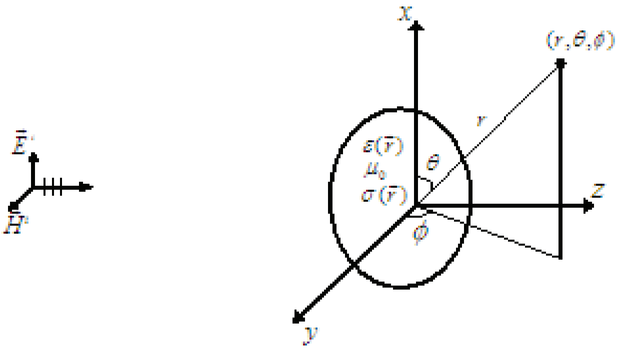

Fig. 1. An arbitrary inhomogeneous dielectric body illuminated by a homogeneous plane wave

The numerical solutions are obtained through the MoM and compared to analytical reference solutions for the special cases of homogeneous and concentric homogeneous spheres. The elements of the MoM matrix are in the form of surface integrals written over the faces of each cube. The required size of the MoM matrix is observed to be quite small providing a rapid convergence.

A time factor $e^{-i\omega t}$ is adopted and suppressed throughout the paper.

2. Formulation

Based on the volume equivalence principle the induction current density induced inside the inhomogeneous lossy dielectric body with arbitrary constitutive parameters $(\varepsilon(\vec{r}),\mu_0,\sigma(\vec{r}))$ under a plane wave incidence (see Fig. 1) is expressed by

$$\vec{J}_{eq}(\vec{r})=\tau(\vec{r})\vec{E}^{tot}(\vec{r}) \quad (1)$$

with

$$\tau(\vec{r}) = i\omega\left[\varepsilon_0 - \varepsilon_c(\vec{r})\right] \tag{2}$$

where (ε_0, μ_0) are the constitutive parameters of free space and $\varepsilon_c(\vec{r})$ is the complex permittivity of the dielectric body. The Green's function technique provides us express the total electric field at any point in space as to the sum of the incident electric field $\vec{E}^i(\vec{r})$ and the scattered field $\vec{E}^s(\vec{r})$ generated by the induction currents in the form

$$\vec{E}^{tot}(\vec{r}) = \vec{E}^i(\vec{r}) + \int_{\vartheta} \tau(\vec{r}')\vec{E}^{tot}(\vec{r}') \cdot \bar{\bar{G}}(\vec{r};\vec{r}')d\vartheta' \tag{3}$$

In (3) $\vec{E}^i(\vec{r}) = \hat{x}e^{ik_0 z}$ with $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$ denoting the free space wavenumber and

$$\bar{\bar{G}}(\vec{r};\vec{r}') = i\omega\mu_0\left(\bar{\bar{I}} + \frac{1}{k_0^2}\nabla\nabla\right)g(\vec{r};\vec{r}') \tag{4}$$

is the well known 3-D dyadic Green's function of free space expressed through the 3-D scalar Green's function of free space

$$g(\vec{r};\vec{r}') = \frac{e^{-ik_0|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \tag{5}$$

with ∇ denoting the standard nabla operator. Due to the double gradiant operation on $g(\vec{r};\vec{r}')$, the dyad $\bar{\bar{G}}(\vec{r};\vec{r}')$ possesses a third order singularity, which renders the volume integral in (3) improper. In what follows we will employ the analytical integral evaluation technique of [7] to remedy this deficiency in the MoM procedure using 3-D pulse basis functions, which requires computing the dyad

$$\bar{\bar{\Omega}}(\vec{r};\vec{r}') = \int_{\vartheta} \bar{\bar{G}}(\vec{r};\vec{r}')d\vartheta' \tag{6}$$

at discrete values of the observation point. The technique introduced in [7] converts the volume integration in (6) into

$$\begin{aligned}\varepsilon_c\,\bar{\bar{\Omega}}(\vec{r};\vec{r}') = -D(\vec{r})\bar{\bar{I}} + \bar{\bar{I}}\nabla \cdot \oint_{\partial\vartheta} g(\vec{r};\vec{r}')\hat{n}(\vec{r}')dS' \\ -\nabla\oint_{\partial\vartheta} g(\vec{r};\vec{r}')\hat{n}(\vec{r}')dS'\end{aligned} \tag{7}$$

with $\hat{n}$ being the outward normal of the enclosure of the region ϑ, denoted by $\partial\vartheta$. The relation (7) can also be expressed in the matrix form

$$\varepsilon_c\,\bar{\bar{\Omega}} = \begin{bmatrix} -D + v_{yy} + v_{zz} & -v_{xy} & -v_{xz} \\ -v_{yx} & -D + v_{xx} + v_{zz} & -v_{yz} \\ -v_{zx} & -v_{zy} & -D + v_{xx} + v_{yy} \end{bmatrix} \tag{8}$$

for the special case of a single cubic element, where

$$D(\vec{r}) = \begin{cases} 1 & \vec{r} \in \vartheta \\ 0 & \vec{r} \notin \vartheta \end{cases} \tag{9}$$

is the characteristic function of the dielectric cube. The rest of the elements of $\bar{\bar{\Omega}}$ are calculated as

$$\begin{aligned}4\pi v_{x_i x_i}(\vec{r}) = \int_{x_{i\oplus1,n}-a}^{x_{i\oplus1,n}+a}\int_{x_{i\oplus2,n}-a}^{x_{i\oplus2,n}+a}\Bigg[\frac{\left(x_i - x_{i,n} - a\right)e^{ik_0 R_{x_i,1}}\left(ik_0 R_{x_i,1} - 1\right)}{R_{x_i,1}^3} \\ - \frac{\left(x_i - x_{i,n} + a\right)e^{ik_0 R_{x_i,2}}\left(ik_0 R_{x_i,2} - 1\right)}{R_{x_i,2}^3}\Bigg]dx'_{i\oplus2}dx'_{i\oplus1}\end{aligned} \tag{10}$$

$$\begin{aligned}4\pi v_{x_i x_{i\oplus1}}(\vec{r}) = \int_{x_{i,n}-a}^{x_{i,n}+a}\int_{x_{i\oplus2,n}-a}^{x_{i\oplus2,n}+a}\left(x_i - x'_i\right)\Bigg[\frac{e^{ik_0 R_{x_{i\oplus1},1}}\left(ik_0 R_{x_{i\oplus1},1} - 1\right)}{R_{x_{i\oplus1},1}^3} \\ - \frac{e^{ik_0 R_{x_{i\oplus1},2}}\left(ik_0 R_{x_{i\oplus1},2} - 1\right)}{R_{x_{i\oplus1},2}^3}\Bigg]dx'_{i\oplus2}dx'_i\end{aligned} \tag{11}$$

$$\begin{aligned}4\pi v_{x_i x_{i\oplus2}}(\vec{r}) = \int_{x_{i,n}-a}^{x_{i,n}+a}\int_{x_{i\oplus1,n}-a}^{x_{i\oplus1,n}+a}\left(x_i - x'_i\right)\Bigg[\frac{e^{ik_0 R_{x_{i\oplus2},1}}\left(ik_0 R_{x_{i\oplus2},1} - 1\right)}{R_{x_{i\oplus2},1}^3} \\ - \frac{e^{ik_0 R_{x_{i\oplus2},2}}\left(ik_0 R_{x_{i\oplus2},2} - 1\right)}{R_{x_{i\oplus2},2}^3}\Bigg]dx'_{i\oplus1}dx'_i\end{aligned} \tag{12}$$

with

$$R_{x_i,1} = \left[\left(x_i - x_{in} - a\right)^2 + \left(x_{(i\oplus1)} - x'_{(i\oplus1)}\right)^2 + \left(x_{(i\oplus2)} - x'_{(i\oplus2)}\right)^2\right]^{1/2} \tag{13}$$

$$R_{x_i,2} = \left[\left(x_i - x_{i,n} + a\right)^2 + \left(x_{i\oplus1} - x'_{i\oplus1}\right)^2 + \left(x_{i\oplus2} - x'_{i\oplus2}\right)^2\right]^{1/2} \tag{14}$$

for $i = 0,1,2$ with (x_0, x_1, x_2) corresponding to (x, y, z) and $(x_{0,n}, x_{1,n}, x_{2,n})$ corresponding to the center of n-th cube. $\oplus$ symbol denotes sum operation in mod 3 and a stands for the length of one side of the cube.

Due to reciprocity, the symmetric nondiagonal matrix elements in (8) require to be equal:

$$v_{x_i x_{i\oplus1}} = v_{x_{i\oplus1} x_i}, \quad i = 0,1,2 \tag{15}$$

$$v_{x_i x_{i\oplus2}} = v_{x_{i\oplus2} x_i}, \quad i = 0,1,2 \tag{16}$$

When the observation point coincides with the source point $\vec{r} = \vec{r}'$, the nondiagonal elements in the matrix in (8) read zero:

$$v_{x_i x_{i\oplus 1}} = v_{x_i x_{i\oplus 2}} = v_{x_{i\oplus 1} x_{i\oplus 2}} = 0, \; i = 0,1,2 \tag{17}$$

in which case the matrix simplifies into

$$\varepsilon_c \overline{\overline{\Omega}} = \begin{bmatrix} -1 + v_{yy}^s + v_{zz}^s & 0 & 0 \\ 0 & -1 + v_{xx}^s + v_{zz}^s & 0 \\ 0 & 0 & -1 + v_{xx}^s + v_{yy}^s \end{bmatrix} \tag{18}$$

with

$$v_{x_i x_i}^s(\vec{r}) = -\frac{a}{2\pi} \int_{-a}^{+a} \int_{-a}^{+a} \left[\frac{e^{-ik_0 R_{x_i}} \left(ik_0 R_{x_i} - 1 \right)}{R_{x_i}^2} \right] dx_{i\oplus 1} dx_{i\oplus 2} \tag{19}$$

and

$$R_{x_i} = \left[a^2 + x_{i\oplus 1}'^2 + x_{i\oplus 2}'^2 \right]^{1/2} \tag{20}$$

for $i = 0,1,2$. Then the regular double integrals in (10)-(12) and (19) and therefore the electrical field generated by a single dielectric cube can be computed effectively.

Next we consider the numerical model of a dielectric sphere constituting cubic elements as depicted in Fig. 2. For evaluating the interaction between two cubic elements with position vectors $\vec{r}_m$ and $\vec{r}_n$ pointing their centers, $\overline{\overline{\Omega}}(\vec{r};\vec{r}')$ in (8) can be written as

$$\varepsilon_c \overline{\overline{\Omega}}^{mn} = \begin{bmatrix} -D + v_{yy} + v_{zz} & -v_{xy} & -v_{xz} \\ -v_{yx} & -D + v_{xx} + v_{zz} & -v_{yz} \\ -v_{zx} & -v_{zy} & -D + v_{xx} + v_{yy} \end{bmatrix} \tag{21}$$

where $\overline{\overline{\Omega}}^{mn} = \overline{\overline{\Omega}}^{mn}(\vec{r}_m;\vec{r}_n)$, $D = D(\vec{r}_m;\vec{r}_n)$ and $v_{x_i x_j} = v_{x_i x_j}(\vec{r}_m;\vec{r}_n)$, $i, j = 0,1,2$. The standard MoM procedure reduces (3) into the linear system of equations

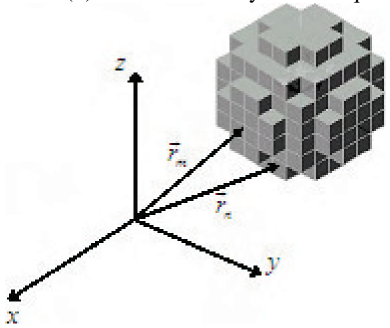

Fig. 2. The dielectric sphere partitioned into cubic elements

$$\sum_{j=1}^{3} \sum_{n=1}^{N} S_{x_i x_j}^{mn} E_{x_j}^{tot}(\vec{r}_n) = -E_{x_i}^{i}(\vec{r}_m) \;,\; m = 1,2,...,N \;,\; n = 1,2,3 \tag{22}$$

with

$$S_{x_i x_j}^{mn} = \tau(\vec{r}_n) \Omega_{x_i x_j}^{mn} - \delta_{ij} \delta_{mn} \tag{23}$$

$S_{x_i x_j}^{mn}$ in (23) has different representations for the following four special cases depending on the relation between indices (i, j) and (m,n):

Case I: $m \neq n$ and $i \neq j$

$$S_{x_i x_j}^{mn} = -\frac{\tau(\vec{r}_n) v_{x_i x_j}(\vec{r}_m;\vec{r}_n)}{\varepsilon_c} \tag{24}$$

Case II: $m \neq n$ and $i = j$

$$S_{x_i x_i}^{mn} = \frac{\tau(\vec{r}_n)}{\varepsilon_c} \left[v_{x_{i\oplus 1} x_{i\oplus 1}}(\vec{r}_m;\vec{r}_n) + v_{x_{i\oplus 2} x_{i\oplus 2}}(\vec{r}_m;\vec{r}_n) \right] \tag{25}$$

Case III: $m = n$ and $i = j$

$$S_{x_i x_i}^{mm} = \frac{-\tau(\vec{r}_n)}{\varepsilon_c} \left[1 - v_{x_{i\oplus 1} x_{i\oplus 1}}^s(\vec{r}_m;\vec{r}_n) - v_{x_{i\oplus 2} x_{i\oplus 2}}^s(\vec{r}_m;\vec{r}_n) \right] - 1 \tag{26}$$

Case IV: $m = n$ and $i \neq j$

$$S_{x_i x_j}^{mm} = 0 \tag{27}$$

In all four cases the reciprocity principle requires

$$S_{x_i x_j}^{mn} = S_{x_j x_i}^{mn} \tag{28}$$

3. Numerical Implementations

In this section we test the numerical scheme the general analytical technique (GAT) presented in Section 2 with the comparison results obtained through Principle Volume Technique (PVT) of [10] and the analytical reference solutions obtained through Mie Series expansion (cf.[13]) for a homogeneous and a coated sphere. The scattering mechanism from a homogeneous dielectric sphere has two distinct characteristics according to the electrical radius of a sphere being much smaller than unity or not. In the former (quasistatic) case $k_0 a << 1$ the total electric field in a homogeneous sphere (see Fig.3) with radius a and relative permittivity $\varepsilon_r = \varepsilon / \varepsilon_0$ yields [8] a static amplitude given by the approximate relation

$$\vec{E}^{tot}(\vec{r}) \cong \frac{3}{(\varepsilon_r + 2)} \vec{E}^{i}(\vec{r}) \tag{29}$$

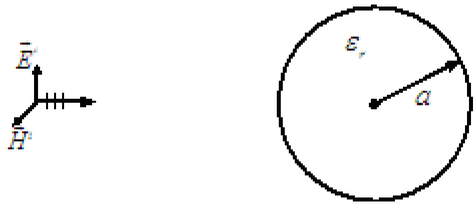

Fig. 3. The dielectric sphere illuminated by a plane wave

An illustration of the accuracy of the numerical schemes corresponding to this case are given in Table 1.

Table 1. Electric field calculated for low frequencies at the center of the homogeneous sphere under plane wave illumination

f [Hz]	ε_r	λ_ε [m]	$3/(\varepsilon_r+2)$	$\lvert \vec{E}^{tot} \rvert$ [V/m] (PVT)	$\lvert \vec{E}^{tot} \rvert$ [V/m] (GAT)
10^7	5.0	$1.342 \cdot 10^1$	0.4286	0.4172	0.4120
10^6	5.0	$1.342 \cdot 10^2$	0.4286	0.4172	0.4120
10^3	5.0	$1.342 \cdot 10^5$	0.4286	0.4172	0.4120
10^3	20.0	$6.708 \cdot 10^4$	0.1364	0.112	0.1398
10^3	51.7	$4.172 \cdot 10^4$	0.0559	0.0503	0.0589

The rotational symmetry of the sphere suffices the calculations to be done only on the quarter of the entire volume. In Fig.s 4-6 we employ 4 cubes to fit along the radius of the sphere so that 70 cubes are used for modeling the geometry, which brings along 210 unknown electric field components. As expected, an increase in the number of cubes improves numerical convergence while at the same time increasing the total computation time regularly.

The relative errors in PVT and GAT results with reference to Mie series solution for the parameterization in Fig. 4 are depicted in Fig. 5.

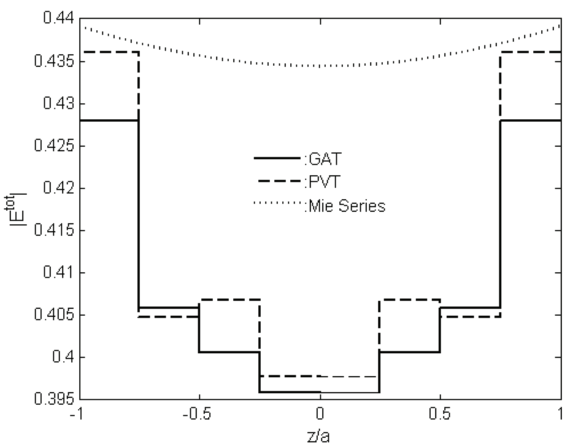

Fig. 4. Total field along z-axis inside a lossless dielectric sphere with $\varepsilon_r = 5$, $k_0 a = 0.1257$.

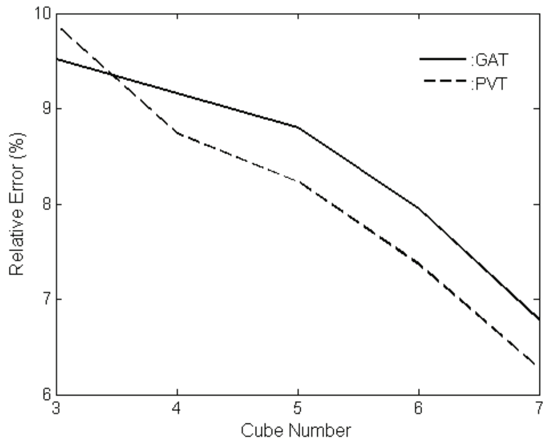

Fig. 5. Relative error for the total field inside a lossless dielectric sphere with $\varepsilon_r = 5$, $k_0 a = 0.1257$.

In Fig. 6 the absorbed power (density) inside the sphere ($\sigma |E|^2/2$) for $\sigma = 0.39$ is depicted for the three methods under the same parameterization.

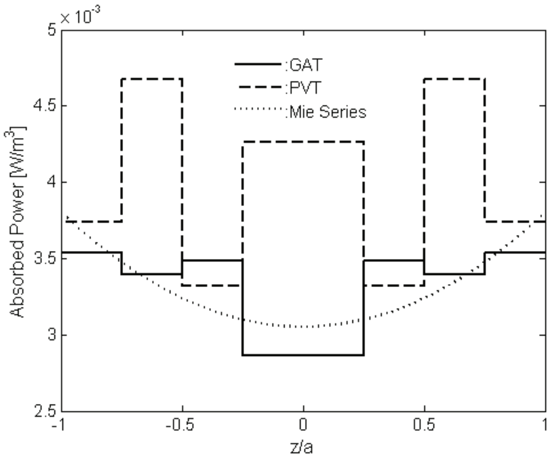

Fig. 6. Absorbed power along z-axis inside dielectric sphere with $\varepsilon_r = 5$, $\sigma = 0.39$, $k_0 a = 0.1257$.

Next we consider the problem of scattering by a concentric homogeneous dielectric sphere as in Fig. 7 with parameterization $\varepsilon_{r1} = 5$, $\varepsilon_{r2} = 16$ and $k_0 a_1 = 0.1257$, $k_0 a_2 = 0.2322$ and again with 4 cubes to fit along the outer radius.

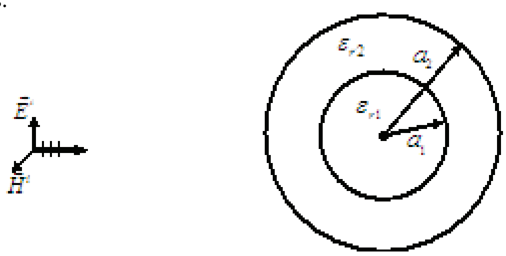

Fig. 7. The coated dielectric sphere illuminated by a plane wave

For this model the fields calculated inside the sphere are depicted in Fig. 8.

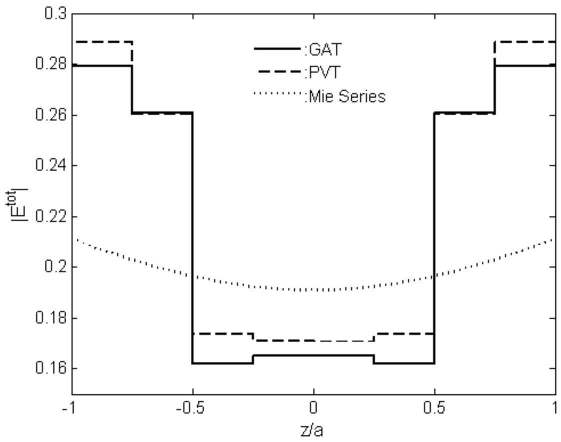

Fig. 8. Electric field along z-axis inside lossless coated dielectric sphere with $\varepsilon_{r1} = 5$, $k_0 a_1 = 0.1257$, $\varepsilon_{r1} = 16$, $k_0 a_2 = 0.2322$

4. Conclusion

The GAT given by [7] for eliminating the singularity of dyadic Green's function is applied to the problem of plane wave scattering by an arbitrary inhomogeneous dielectric body. The main feature of GAT is to avoid any exclusion volume and operations for removing the singularity of dyadic Green's function in source region which makes its application easier in practice. Calculations are compared to analytical expressions for the special cases of homogeneous and concentric homogeneous spheres under plane wave incidence. The total scattered field inside the objects and relative error in calculations as compared to analytical Mie series solutions are provided. In each case rapid convergence is observed with a relatively small number of cubes, while the developed GAT algorithms converge much faster than the PVT algorithms as expected theoretically. The numerical efficiency (rapid convergence and relative error) of GAT algorithms provide a reliable option for extending the methodology to the investigation of scattering by dielectric bodies of arbitrary shape.

5. References

[1] C.T. Tai, "Dyadic Green's Functions in Electromagnetic Theory", IEEE Press, 1971.

[2] J. Van Bladel, "Some Remarks on Green's Dyadic for Infinite Space", *IRE Trans. Antennas Propagat.* AP-9, 1961, pp. 563-566.

[3] A.D. Yaghjian, "Electric Dyadic Green's Functions in the Source Region", *Proc. IEEE* 68, No.2, pp. 248-263, 1980.

[4] A.D. Yaghjian, "A Delta-Distribution Derivation of Electric Field in the Source Region", *Electromagnetics* 2, pp. 161-167, 1982.

[5] I.M. Gel'fand and G.E. Shilov, "Generalized Functions", Academic, New York, 1964, pp. 295-305.

[6] S.W. Lee, J. Boersma, C.L. Law and G. A. Deschamps, "Singularity in Green's Function and Its Numerical Evaluation", *IEEE. Trans. Antennas Propagat*. AP-28, No.3, pp. 311-317, 1980.

[7] G. Gao, C.T. Verdin and T.M. Habashy, "Analytical Techniques to Evaluate the Integrals of 3D and 2D Spatial Dyadic Green's Functions", *PIER* 52, pp. 47-80, 2005.

[8] J.H. Richmond. "Scattering by Dielectric Cylinder of Arbitrary Cross Section Shape", *IEEE Trans. Antennas Propagat.* AP-35, pp. 334-341, 1965.

[9] R.F. Harrington, "Field Computation by Moment Methods", Macmillan, New York, 1968.

[10] D.E. Livesay and K. Chen, "Electromagnetic Fields Induced Inside Arbitrarily Shaped Biological Bodies", *IEEE Trans. Microwave Theory Tech.* MTT-22, No. 12, pp. 1273-1280, 1974.

[11] D.H. Schaubert, D.R. Wilton and A.W. Glisson, "A Tetrahedral Modeling Method for Electromagnetic Scattering by Arbitrarily Shaped Inhomogeneous Dielectric Bodies", *IEEE Trans.Antennas Propagat.* AP-32, No. 1, pp. 77-85, 1984.

[12] K. Sertel and J.L. Volakis, "Method of Moments Solution of Volume Integral Equations Using Parametric Geometry Modeling", *Radio Science* 37 (10), No.1, pp. 1-7, 2002.

[13] S. Rukspollmuang and K-M. Chen, "Heating of Spherical Versus Realistic Models of Human and Infrahuman Heads by Electromagnetic Waves", *Radio Sci*ence 14 (68), pp. 51-62, 1979.