Frequency-Domain Steady-State Analysis of Circuits with Mem-Elements

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Abstract

The paper deals with frequency-domain models of memristor, memcapacitor, and meminductor, and their use in the steady-state analysis by means of the harmonic-balance method. The models are based on a polynomial approximation of constitutive relations, allowing analytical formulation of relations between the spectral components of stimulus and response, for both periodic and quasi-periodic steady-state conditions. It is not necessary to transform signals between the time and the frequency domains to obtain the mem-element response. Example analyses demonstrate the model use.

1. Introduction

The steady-state analysis represents an important tool for the characterization of electrical circuits. In the case of linear time-invariant circuits, the harmonic steady state can be obtained using the Heaviside operator method, which is well-known as the AC analysis. The purpose of the paper is to present simple analytical frequency-domain models of mem-elements, which are suitable for steady-state analysis by means of the Harmonic Balance method [1].

The fabrication of memristor in 2008 in the form of a nanoscale device by HP laboratories [2] triggered a wave of interest in mem-systems. As the HP memristor is still not available for researchers, any verification of proposed circuits is based on simulations or emulations [3], [4].

A number of models of memristor have been proposed so far; see [5], [6], [7] and the references therein. The models are more or less based on the HP memristor, which is not an ideal element but a generalized memristive system [8].

In addition to memristive systems, there are memcapacitative and meminductive systems as well as their special subsystems, namely memcapacitors and meminductors [9], [10].

An entirely new approach to modeling was proposed in [11] and [12]. The proposed models are based on explicit constitutive relations of *ideal* mem-elements.

For example, the two possible constitutive relations of ideal memristor have the form

$$\varphi = \varphi(q) \ \text{ and } \ q = q(\varphi) , \tag{1a,b}$$

where

$$\varphi = \int_{-\infty}^{t} v(\tau)d\tau , \ q = \int_{-\infty}^{t} i(\tau)d\tau \tag{2a,b}$$

are the flux and the charge, respectively, while $\varphi(q)$ and $q(\varphi)$ are the nonlinear constitutive functions for *charge*- and *flux*-controlled memristors, respectively.

Memristor terminal quantities are then [11]

$$v(t) = \frac{d\varphi(q)}{dt} = \frac{d\varphi(q)}{dq}\frac{dq}{dt} = R_M(q)i(t) , \tag{3a}$$

$$i(t) = \frac{dq(\varphi)}{dt} = \frac{dq(\varphi)}{d\varphi}\frac{d\varphi}{dt} = G_M(\varphi)v(t) , \tag{3b}$$

where $R_M(q) = \frac{d\varphi(q)}{dq}$ and $G_M(\varphi) = \frac{dq(\varphi)}{d\varphi}$ are the memresistance and the memconductance, respectively.

The method of harmonic balance (HB) [1] is suitable for cases where the waveforms of all circuit quantities can be approximated by a relatively small number of harmonic components $X^{(k)}$, i.e. by

$$x(t) = \sum_{k=-N}^{N} X^{(k)} e^{j\omega_k t} , \tag{4}$$

where the set of frequencies ω_k is chosen before the analysis.

The port voltage-current relations of individual network elements have to be transformed into the operator domain, assigning a vector of spectral components of response quantity to a given vector of stimulus. It is a trivial task for linear circuit elements. However, it requires a rather complicated processing in the case of nonlinear components including transformations between the frequency and the time domains [13].

Although the memristor is a nonlinear element, the model (1)-(3) allows an explicit calculation of spectral components of the response variable in case constitutive relation (1) is expressed as a polynomial.

Section 2 of the paper deals with deriving frequency-domain models of mem-elements, and section 3 gives a numerical example.

2. Frequency-Domain Models of Mem-Elements

2.1. Memristor

In the case of the charge-controlled memristor, terminal voltage v can be calculated from the independent terminal current i (stimulus) as follows

$$i(t) \rightarrow q(t) \rightarrow R_M(q) \rightarrow v(t) = i(t)\, R_M(q) , \tag{5}$$

and vice versa for the flux-controlled memristor

$$v(t) \to \varphi(t) \to G_M(\varphi) \to i(t) = v(t)\, G_M(\varphi)\ . \quad (6)$$

Let us suppose that the constitutive relation (1a) of a charge-controlled memristor is expressed in the polynomial form as

$$\varphi(q) = \sum_{k=1}^{M} r_k q^k\ , \quad (7)$$

where r_k are the polynomial coefficients.

Let us further suppose that the system under analysis is in the periodic steady state with one fundamental frequency ω_0, and that the time domain waveforms are approximated by N harmonics. Then the independent terminal current i can be expressed as

$$i(t) = \sum_{k=-N}^{N} I^{(k)} e^{jk\omega_0 t}\ , \quad (8)$$

where $I^{(k)}$ are the complex spectral components.

The charge is then

$$q(t) = \int_{-\infty}^{t} i(\tau) d\tau = \sum_{\substack{k\neq 0 \\ k=-N}}^{N} \frac{I^{(k)}}{jk\omega_0} e^{jk\omega_0 t} + Q^{(0)}\ . \quad (9)$$

As can be seen, (9) imposes a restriction on the DC component of current, $I^{(0)} = 0$, and introduces the DC component of charge $Q^{(0)}$ as another unknown variable. Thus the set of equations of the harmonic balance method for the whole circuit will be extended by one equation and one unknown for each memristor. The condition $I^{(0)} = 0$ can be seen as a necessary condition for the existence of steady state.

The next step consists in the calculation of memresistance (3a). With respect to constitutive relation (7), R_M will be again a polynomial of q

$$R_M(q) = \sum_{k=1}^{M} k r_k q^{k-1}\ . \quad (10)$$

Let us consider the square of charge

$$q^2(t) = \left(\sum_{k=-N}^{N} Q^{(k)} e^{jk\omega_0 t} \right)^2 = \sum_{k=-2N}^{2N} \left(\sum_{j=\max(-N,-N+k)}^{\min(N,N+k)} Q^{(j)} Q^{(k-j)} \right) e^{jk\omega_0 t} \quad (11)$$

where $Q^{(k)}$ are the spectral components. After performing (11), the number of harmonics is doubled. The operation is equivalent to the (full) convolution of vectors of spectral components $\mathbf{Q}$. Formally, the operator of multiplication in (10) will be replaced by the operator of convolution. The spectrum of $\mathbf{R}_M$ will be

$$\mathbf{R}_M = \sum_{k=1}^{M} k r_k \left((*)^{k-1} \mathbf{Q} \right)\ . \quad (12)$$

The number of harmonics of R_M depends on the degree of polynomial (10). Finally, the vector of harmonic components of the terminal voltage will be

$$\mathbf{V} = \mathbf{I} * \mathbf{R}_M\ . \quad (13)$$

Since the number of spectral components of the terminal voltage should be the same as the number of components of the terminal current, vector $\mathbf{V}$ should be truncated into the interval $-N,..., +N$.

The model of a flux-controlled memristor can be derived in the same way. In the multi-tone case, where frequencies ω_k in (4) are not integer multiples of the fundamental frequency, the procedure for the calculation of frequency-domain response will be similar. With respect to the mapping of spectral components into a one-dimensional array, (12) and (13) will not be expressed in terms of the convolution operator.

2.2. Memcapacitor

The constitutive relation of ideal voltage-controlled memcapacitor is

$$\sigma = \sigma(\varphi)\ , \quad (14)$$

where

$$\sigma = \int_{-\infty}^{t} q(\tau) d\tau \quad (15)$$

is the integral of charge [10].

The memcapacitor charge and current are then

$$q(t) = \frac{d\sigma(\varphi)}{dt} = \frac{d\sigma(\varphi)}{d\varphi}\frac{d\varphi}{dt} = C_M(\varphi) v(t) \quad (16)$$

$$i(t) = \frac{d}{dt}\left(C_M(\varphi) v(t) \right)\ , \quad (17)$$

where $C_M(\varphi) = \dfrac{d\sigma(\varphi)}{d\varphi}$ is the memcapacitance.

The terminal current i can be calculated from the independent terminal voltage v (stimulus) as follows

$$v(t) \to \varphi(t) \to C_M(\varphi) \to i(t) = d/dt\, (v(t)\, C_M(\varphi)). \quad (18)$$

The spectrum of flux Φ can be calculated similarly to (9), which imposes the restriction $V^{(0)} = 0$, and introduces the DC component of flux $\Phi^{(0)}$ as another unknown variable.

Similarly to (7) and (10), constitutive relation (14) will be assumed in the polynomial form with coefficients c_k. Then the spectrum of memcapacitance will be

$$\mathbf{C}_M = \sum_{k=1}^{M} k c_k \left((*)^{k-1} \Phi \right)\ . \quad (19)$$

Finally, the vector of harmonic components of the terminal current will be

$$\mathbf{I} = \Delta(\mathbf{V} * \mathbf{C}_M)\ , \quad (20)$$

where Δ is the operator of differentiation in the frequency domain, i.e. each spectral component is multiplied by $j\omega_k$.

The constitutive relation of ideal charge-controlled memcapacitor is [10]

$$\varphi = \varphi(\sigma) \,. \qquad (21)$$

The memcapacitor voltage is then

$$v(t) = \frac{d\varphi(\sigma)}{dt} = \frac{d\varphi(\sigma)}{d\sigma}\frac{d\sigma}{dt} = D_M(\sigma)q(t) \,, \qquad (22)$$

where $D_M(\sigma) = \dfrac{d\varphi(\sigma)}{d\sigma}$ is the inverse memcapacitance.

The terminal voltage v can be calculated from the independent terminal current i (stimulus) as follows

$$i(t) \rightarrow q(t) \rightarrow \sigma(t) \rightarrow D_M(\sigma) \rightarrow v(t) = q(t)\, D_M(\sigma). \qquad (23)$$

The spectra of charge and its integral can be calculated similarly to (9). There will be two restrictive conditions, $I^{(0)} = 0$ and $Q^{(0)} = 0$, and the DC component of the integral of charge $S^{(0)}$ will be introduced as another unknown variable.

Similarly to (7) and (10), constitutive relation (21) will be assumed in the polynomial form with coefficients d_k. Then the spectrum of inverse memcapacitance will be

$$\mathbf{D}_M = \sum_{k=1}^{M} k d_k \left((*)^{k-1} \mathbf{S} \right) \,. \qquad (24)$$

Finally, the vector of harmonic components of the terminal voltage will be

$$\mathbf{V} = \mathbf{Q} * \mathbf{D}_M \,. \qquad (25)$$

2.3. Meminductor

The constitutive relation of ideal current-controlled meminductor is

$$\rho = \rho(q) \,, \qquad (26)$$

where

$$\rho = \int_{-\infty}^{t} \varphi(\tau) d\tau \qquad (27)$$

is the integral of flux [10].

The meminductor flux and voltage are then

$$\varphi(t) = \frac{d\rho(q)}{dt} = \frac{d\rho(q)}{dq}\frac{dq}{dt} = L_M(q) i(t) \qquad (28)$$

$$v(t) = \frac{d}{dt}\left(L_M(q) i(t) \right), \qquad (29)$$

where $L_M(q) = \dfrac{d\rho(q)}{dq}$ is the meminductance.

The terminal voltage v can be calculated from the independent terminal current i (stimulus) as follows

$$i(t) \rightarrow q(t) \rightarrow L_M(q) \rightarrow v(t) = d/dt\, (i(t)\, L_M(q)). \qquad (30)$$

Similarly to (19), the spectrum of meminductance will be

$$L_M = \sum_{k=1}^{M} k l_k \left((*)^{k-1} Q \right) \,. \qquad (31)$$

The vector of harmonic components of the terminal current will be

$$\mathbf{V} = \Delta(\mathbf{I} * \mathbf{L}_M) \,. \qquad (32)$$

The constitutive relation of ideal flux-controlled meminductor is [10]

$$q = q(\rho) \,. \qquad (33)$$

The meminductor current is then

$$i(t) = \frac{dq(\rho)}{dt} = \frac{dq(\rho)}{d\rho}\frac{d\rho}{dt} = \Lambda_M(\rho)\varphi(t) \qquad (34)$$

where $\Lambda_M(\rho) = \dfrac{dq(\rho)}{d\rho}$ is the inverse meminductance.

The terminal current i can be calculated from the independent terminal voltage v (stimulus) as follows

$$v(t) \rightarrow \varphi(t) \rightarrow \rho(t) \rightarrow \Lambda_M(\rho) \rightarrow i(t) = \varphi(t)\, \Lambda_M(\rho). \qquad (35)$$

Similarly to (9), there will be two restrictive conditions, $V^{(0)} = 0$ and $\Phi^{(0)} = 0$, and the DC component of the integral of flux $F^{(0)}$ will be introduced as another unknown variable.

Then the spectrum of inverse meminductance will be

$$\mathbf{\Lambda}_M = \sum_{k=1}^{M} k\lambda_k \left((*)^{k-1} \mathbf{F} \right) \qquad (36)$$

and the vector of harmonic components of the terminal current will be

$$\mathbf{I} = \Phi * \mathbf{\Lambda}_M \,. \qquad (37)$$

3. Linear Circuit with Mem-Element

Let us consider a connection of a linear circuit, which may include independent sources, with one mem-element, Fig. 1.

For current/charge-controlled mem-elements, the linear part can be replaced by the Thévenin equivalent circuit represented by $(2N+1)$ spectral components of equivalent voltage $V^{(k)}$ and series impedances $Z^{(k)}$. KVL for the loop in Fig. 1a can be written as

$$V^{(k)} - Z^{(k)} I_M^{(k)} - V_M^{(k)}(\mathbf{I}_M) = 0 \,, \; k = -N, ..., N, \qquad (38)$$

where $V_M^{(k)}(\mathbf{I}_M)$ is the k-th spectral component of mem-element voltage, which is a function of all spectral components of current $\mathbf{I}_M$.

For voltage/flux-controlled mem-elements, the Norton equivalent circuit in Fig. 1b gives

$$I^{(k)} - Y^{(k)} V_M^{(k)} - I_M^{(k)}(\mathbf{V}_M) = 0 \,, \; k = -N, ..., N. \qquad (39)$$

If the memristor is connected to a linear circuit, there is no means how to determine the mem-element internal state by a linear observer in order to set or stabilize the DC components of integrals (see Section II). Thus the necessary condition for the existence of the steady state, $I_M^{(0)} = 0$ or $V_M^{(0)} = 0$, should be fulfilled by design. Then the DC components can be arbitrary, i.e. the system has an *infinite number of steady states*.

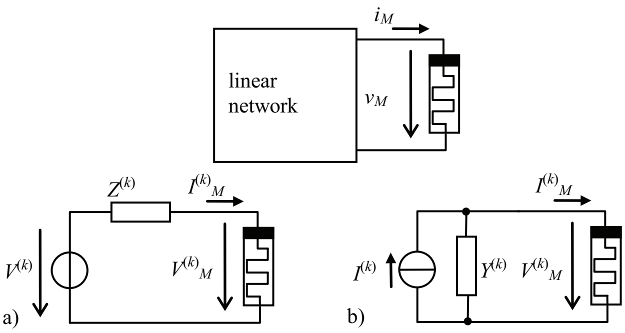

Fig. 1. Linear circuit with mem-element and its Thévenin (a) and Norton (b) equivalent circuit for individual spectral components.

With respect to the symmetry of spectral components, (38) and (39) represent N complex equations for N complex unknowns (or equivalently $2N$ real equations for $2N$ real unknowns) in the implicit form

$$\mathbf{F}(\mathbf{I}_M) = 0 \text{ or } \mathbf{F}(\mathbf{V}_M) = 0 . \quad (40)$$

The solution of (40) represents an approximation of the steady state by an a priori chosen number N of harmonics. The initial guess may be given by the AC solution for $I^{(1)}$ or $V^{(1)}$.

Let us consider the single-tone circuit in Fig. 2 with current-controlled meminductor. The meminductor constitutive relation (26), (31) is characterized by two polynomial coefficients, $l_1 = 0.1$ H and $l_3 = 10^3$ HC^{-2}. The necessary condition $I_M^{(0)} = 0$ is fulfilled by design. Parameters were chosen such that the resonance occurs at 15.9 Hz, considering only the linear term l_1 of (26).

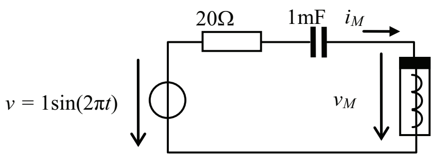

Fig. 2. Example of linear circuit with current-controlled meminductor.

The method of harmonic balance was experimentally implemented in Matlab. Equation (40) was solved using the *fsolve* function. Figure 3 shows the steady-state amplitude of $I^{(1)}$ for different values of $Q^{(0)}$. It was calculated for the number of harmonics $N = 10$. It is evident that the resonance frequency depends on the DC component of charge.

Figure 4 shows the steady-state waveforms of meminductor voltage for $Q^{(0)} = 10$ mC for frequencies of 2 Hz and 8 Hz.

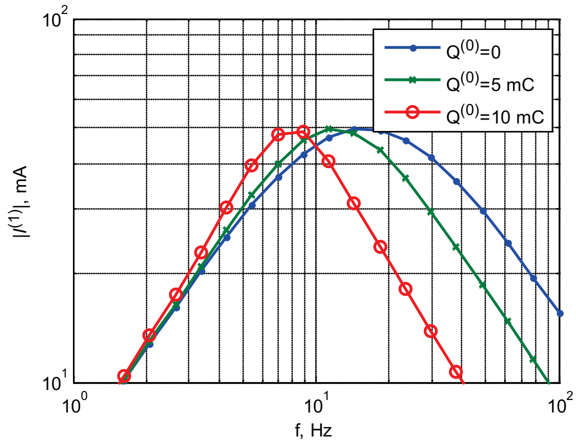

Fig. 3. Resonance curves of $I^{(1)}$ for different values of $Q^{(0)}$.

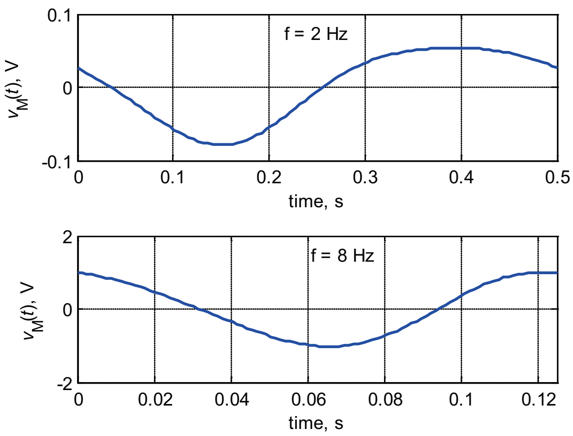

Fig. 4. Time-domain waveforms of $v_M(t)$ for $Q^{(0)} = 10$mC.

4. Conclusions

The frequency-domain models presented in the paper allow formulating analytically the relations between the spectral components of stimulus and response, both for periodic and quasi-periodic steady-state conditions. It is not necessary to transform signals between the time and the frequency domains to obtain the mem-element response.

Acknowledgment

This work has been supported by the Czech Science Foundation under grant No P102/10/1614, the Czech Ministry of Education under research project No MSM0021630513, by the project CZ.1.07/2.3.00/20.0007 WICOMT of the operational programme Education for competitiveness, and Project for the development of K217 Department, UD Brno – Modern electrical elements and systems.

The research leading to these results has received funding from the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement No 230126.

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