Green`s Function Approach to The Continuity Equation

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***Abstract*—Minority carriers continuity equation has been studied by Green's function in rectangular coordinate system. The exact solution of this differential equation, both time and space dependent has been obtained by Green's function approach. Calculation has been done for the *p*-region of a *p-n* junction under the low-level injection and quasineutrality assumptions. Stimulation of carriers has been done by an impulse of light. Green's function approach gives an opportunity to envisage each nonhomogeneous boundary and initial value contribution to the device performance. The solution shows a clear picture of the impact of physical parameters on device performance. The method is more convenient for comparing different devices' performance, particularly in case of light - semiconductor interaction.**

I. INTRODUCTION

Conventional solution of one dimensional continuity equation results in a combination of hyperbolic functions, which are very much different in behaviour [1]. In some cases, it is hard to interpret the solution and extract the design parameters. Even in case of nonhomogeneous boundary conditions the solution itself is more difficult to get [2]. Generally, with some simplifying assumptions, approximate solutions are used.

The Green`s function approach gives exact solution of the differential equation in integral form. Contribution of generation function, boundary and initial values are separate. The main difficulty of the method is construction of Green`s function which requires solution of homogeneous version of the problem. Once the Green`s function of the problem has been constructed, the solution of any kind of generation function, nonhomogeneous boundaries, and initial value are the matter of some integral taken. A drawback of the method is that the solution has a serial form. The convergence of series is dependent on boundary values. In this particular problem, it rapidly converges. Green`s function solution of the continuity equation gives more elegant presentation of device properties. It is easy to interpret, and to see the impact of physical parameters on device performance.

One dimensional continuity equation has the same mathematical form as that of heat conduction problem except for the recombination term. The heat conduction problem has been well-studied in the literature[3]. The solutions of various problems have been tabulated for ready use. To utilise this opportunity, the continuity equation has been transformed to the heat conduction problem. After appropriate solution of the heat conduction problem by Green`s function, an inverse transformation has been invoked to get minority carrier population throughout the device.

In this paper, the Green's function approach to the heat conduction problem has been reviewed, first. Then, the continuity equation of minority carriers has been transformed to the heat conduction equation. Following that, the Green`s function of the transformed problem has been constructed. The solution of transformed problem has been calculated for specific boundary values. Finally, the minority carriers population throughout the device has been calculated by using inverse transformation.

II. REVIEW OF HEAT CONDUCTION PROBLEM

The heat conduction problem can be described as finding out variation of temperature in a solid. The solution of the differential equation under the various boundary and initial values has been presented in one-dimensional rectangular coordinate system[3].

A general nonhomogeneous heat conduction differential equation with accompanied boundary and initial values is as follows:

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{1}{k} g(u,t) = \frac{1}{\alpha} \frac{\partial \psi}{\partial t} \qquad (1)$$

$$u = 0 \;, \quad k_1 \frac{\partial \psi}{\partial u} + h_1 \psi = f_1(t) \;, \quad t > 0,$$

$$u = L \;, \quad k_2 \frac{\partial \psi}{\partial u} + h_2 \psi = f_2(t) \;, \quad t > 0,$$

$$t = 0 \;, \quad \psi = F(u)$$

Where ψ is temperature; k and α are constants of differential equation; k_1, k_2, h_1, and h_2 are constant for convenient boundary conditions; $f_1(t)$, $f_2(t)$ are nonhomogeneous boundary functions; $F(u)$ is the initial value function; $g(u,t)$ is the generation function.

The Green`s function approach to the solution is as follows:

for $t > \tau$,

$$\psi(u,t) = \int_{v=0}^{L} G(u,t|v,\tau)\big|_{\tau=0} F(v)dv$$

$$+\frac{\alpha}{k} \int_{\tau=0}^{t} d\tau \int_{v=0}^{L} G(u,t|v,\tau) g(v,\tau)dv$$

$$+\alpha \int_{\tau=0}^{t} G(u,t|v,\tau)|_{v=0} \frac{1}{k_1} f_1(\tau)d\tau$$

$$+\alpha \int_{\tau=0}^{t} G(u,t|v,\tau)|_{v=L} \frac{1}{k_2} f_2(\tau)d\tau$$

for $t > \tau$, $G(u,t|v,\tau) = 0$ (2)

The Green`s function, $G(u,t|v,\tau)$, presents the temperature at position u and time t in a solid in case of a heat source of impulse has been situated at position v and time τ. Also, boundary values are homogeneous. $\psi(u,t)$ is total temperature variation in the solid at position u and time t under the original generation function, initial and boundary values. The first term is the contribution from initial value of $F(u)$; the second term is the contribution from generation function of $g(u,t)$; last two terms are contributions from nonhomogeneous boundaries. Once the Green`s function of the problem has been constructed, the solution for any arbitrary generation function, and any arbitrary nonhomogeneous boundary value can be calculated by integration. In order to avoid the replication, the construction of the Green`s function has been done according to the specific boundary conditions of particular continuity problem in the following subsections.

III. SOLUTION OF CONTINUITY EQUATION

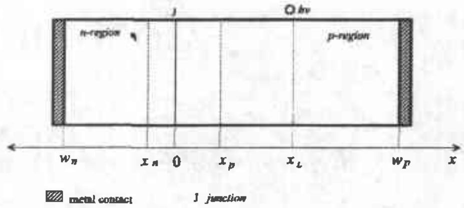

Fig.1 Schematic presentation of a planer *p-n* junction.

One can construct the following well-known continuity equation for minority carriers. The problem has been posed in the *p-region* of a *p-n junction* bulk semiconductor device (Fig.1) with suitable boundary conditions. The quasineutrality and low-level injection have been assumed[1,2].

$$\frac{\partial^2 n}{\partial x^2} - \frac{n}{L_n^2} + \frac{g_L}{D_n} = \frac{1}{D_n} \frac{\partial n}{\partial t} \qquad (3)$$

$$x = x_p \qquad n \equiv f_{n1}(t) = 0 \;, \quad t>0$$

$$x = w_p \qquad n \equiv f_{n2}(t) = 0 \;, \quad t>0$$

$$t = 0 \;, \qquad n = F_n(x) = 0 \;,$$

$$L_n^2 = D_n \, \tau_n$$

where, L_n is diffusion length; D_n is diffusion constants; τ_n is recombination time constant; n is excess minority carrier concentrations (1/cm^{-3}); g_L is generation function.

One dimensional (1D) device structure is shown in the Fig.1. The problem has been transported to the new more convenient coordinate system by changing the variable for *p*-region.

$$u \equiv x - x_p \;, \quad x = x_p \Rightarrow u = 0$$

$$x = w_p \Rightarrow u = w_p - x_p \equiv L$$

$$du = dx \qquad (4)$$

$$x_p \le x \le w_p \quad \Rightarrow 0 \le u \le L$$

According to the new defined independent variable, the differential equation reads

$$\frac{\partial^2 n}{\partial u^2} - \frac{n}{L_n^2} + \frac{g_L}{D_n} = \frac{1}{D_n} \frac{\partial n}{\partial t} \qquad (5)$$

Except recombination term, the differential equation has same mathematical form as that of heat conduction problem. In order to get rid of recombination term, the following transformation should be carried out[4]:

$$\psi_n(u,t) = n(u,t)\, e^{t/\tau_n} \qquad (6)$$

The continuity equation has been transformed to the heat conduction equation.

$$\frac{\partial^2 \psi_n}{\partial u^2} + \frac{e^{(t/\tau_n)}}{D_n} g_L = \frac{1}{D_n}\frac{\partial \psi_n}{\partial t} \qquad (7)$$

$u = 0$, $\psi_n = f_{n1}(t) = 0$, $t > 0$

$u = L$, $\psi_n = f_{n2}(t) = 0$, $t > 0$

$t = 0$, $\psi_n = F_n(u) = 0$

The last differential equation with accompanied boundary values constitute the problem which is subject to Green`s function solution. Both Eq.1 and Eq.7 have the same cast form. So, the solution of the both should be same mathematically. The main task, here is to construct the related Green`s function. For that reason, the generation function should have the following form.

$$g_L(u) = M\delta(u - u_L) \qquad (8)$$

where, $u_L \equiv x_L - x_p$

In other words, M electron-hole pairs per cm^{-3} per second have been injected at the position of $u = u_L$.

First of all, the homogenous problem should be considered.

$$\frac{\partial^2 \psi_n}{\partial u^2} = \frac{1}{D_n}\frac{\partial \psi_n}{\partial t} \qquad (9)$$

$u = 0$, $\psi_n = f_{n1}(t) = 0$, $t > 0$

$u = L$, $\psi_n = f_{n2}(t) = 0$, $t > 0$

$t = 0$, $\psi_n = F_n(u) = 0$

For the solution, one can apply the technique of separation of variables [3,5].

$$\psi_n(u,t) \equiv T_n(t) U_n(u) \qquad (10)$$

By the end, one gets the following ordinary differential equations with separation constant, β_m.

$$\frac{1}{U_n}\frac{d^2 U_n}{du^2} = \frac{1}{D_n}\frac{1}{T_n}\frac{dT_n}{dt} = -\beta_m^2 \qquad (11)$$

$$\frac{dT_n}{dt} + D_n \beta_m^2 T_n = 0 \qquad (12)$$

$$\frac{d^2 U_n}{du^2} + \beta_m^2 U_n = 0 \qquad (13)$$

$u = 0$, $U_n = 0$

$u = L$, $U_n = 0$

The solution of time dependent ordinary differential equation is straightforward.

$$T_n(t) = e^{-D_n \beta_m^2 t} \qquad (14)$$

The second ordinary differential equation with accompanied boundary conditions form *special Sturm-Liouville problem*, which has the following solution[5].

$$U_n(\beta_m, u) = Sin\beta_m u \qquad (15)$$

Combining both solutions, one obtains,

$$\psi_n(u,t) = \sum_{m=1}^{\infty} C_m U_n(\beta_m, u) e^{-D_p \beta_m^2 t} \qquad (16)$$

$$C_m = \frac{1}{N(\beta_m)} \int_0^L U_n(\beta_m, v) F_n(v) dv \qquad (17)$$

$$\frac{1}{N(\beta_m)} = \frac{2}{L} \qquad (18)$$

$$\beta_m = m\frac{\pi}{L} \ , \quad m = 1,2,3,... \qquad (19)$$

β_m is zero of $Sin(\beta_m L) = 0$

β_m and $U(\beta_m, u)$ are eigenvalue and eigenfunction of *Special Sturm-Liouville problem,* respectively. Coefficients C_m can be determined by the orthogonal property of the eigenfunctions. $N(\beta_m)$ is the normalising factor of eigenfunctions.

After appropriate substitutions:

$$\psi_n(u,t) = \frac{2}{L}\sum_{m=1}^{\infty} e^{-D_n \beta_m^2 t} Sin(\beta_m u) \times \int_o^L Sin(\beta_m v) F_n(v) dv \qquad (20)$$

v is a dummy variable has been introduced for calculation of integral.

In order to extract Green's function, the solution should be rearranged in its cast form[3].

$$\psi_n(u,t) = \int_{v=0}^{L} G_n(u,t|v,\tau)\Big|_{\tau=0} F_n(v)dv \qquad (21)$$

$$\psi_n(u,t) = \int_{v=0}^{L} \left[\frac{2}{L} \sum_{m=1}^{\infty} e^{-D_n \beta_m^2 t} Sin(\beta_m u) Sin(\beta_m v) \right] \times F_n(v)dv \qquad (22)$$

The term in square brackets is Green's function of the transformed problem at $\tau = 0$. For any τ, t should be replaced with *(t-τ)* in above Green's function[3].

for $t > \tau$,

$$G_n(u,t|v,\tau) = \frac{2}{L} \times \sum_{m=1}^{\infty} e^{-D_n \beta_m^2 (t-\tau)} Sin(\beta_m u) Sin(\beta_m v) \qquad (23)$$

for $t < \tau$, $G(u,t|v,\tau)=0$

This is the Green's function of the transformed problem for any $t > \tau$. When Eq.1 type solution is considered, since the boundary values (f_1 and f_2) and initial value ($F(v)$) are zero, the only term that should be considered is the generation term.

$$\psi_n(u,t) = \frac{D_n}{k} \int_{\tau=0}^{t} d\tau \int_{v=0}^{L} G_n(u,t|v,\tau) \times \frac{e^{t/\tau_n}}{D_n} g_L(v,\tau)dv \qquad (24)$$

$k=1$

For an impulse type generation function,

$$\psi_n(u,t) = D_n \int_0^t d\tau \int_{v=0}^{L} \frac{2}{L} \sum_{m=1}^{\infty} e^{-D_n \beta_m^2 (t-\tau)} \times Sin(\beta_m u) Sin(\beta_m v) \frac{e^{\tau/\tau_n}}{D_n} M\delta(v-u_L)dv \qquad (25)$$

After carrying out the integration, one obtains the following solution of transformed problem.

$$\psi_n(u,t) = M \frac{2}{L} \sum_{m=1}^{\infty} \frac{\tau_n}{(L_n \beta_m)^2 + 1} (e^{t/\tau_n} - e^{-D_n \beta_m^2 t}) \times Sin(\beta_m u) Sin(\beta_m u_L) \qquad (26)$$

Using inverse transformation,

$$n = \psi_n \, e^{-t/\tau_n} \qquad (27)$$

After returning to the original coordinate system, and substituting related parameters,
$u = x - x_p$, $u_L = x_L - x_p$,

$\beta_m = m\frac{\pi}{L}$, $m = 1,2,3,...$

one obtains:

$$n(x,t) = M \frac{2}{L} \sum_{m=1}^{\infty} \frac{\tau_n}{(L_n \beta_m)^2 + 1} (1 - e^{-\frac{(L_n \beta_m)^2 + 1}{\tau_n} t}) \times Sin[\beta_m (x - x_p)] Sin[\beta_m (x_L - x_p)] \qquad (28)$$

As a result of this calculation, this describes the electron population in the *p-region* of a *p-n* junction in terms of time and space, under the injection of M electron-hole pairs per cm^{-3} per second. In case of any arbitrary generation function and nonhomogeneous boundary conditions, it is only matter of some integrals-taken. Any integration difficulties can be overcome at least by numerical integration.

TABLE I
SIMULATION DATA[6].

Number of injected carriers, M[1/cm^3]	10^{14}
Doping concentration, N_A [1/cm^3]	5.0 10^{17}
Recombination time constant, τ_n [s]	5.0 10^{-10}
Depletion edge in *p*-region, x_p[cm]	0.1 10^{-4}
p-region length, w_p [cm]	1.1 10^{-4}
Light source placement, x_L [cm]	0.6 10^{-4}

Electron population for a *GaAs p-n* junction with tabulated properties is given in the Fig.2.

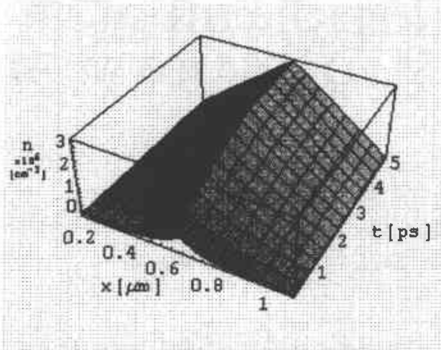

Fig.2 Both time (t) and space (x) dependent electron population (n) in the p-region of a $GaAs$ p-n junction.

III.DISCUSSION OF RESULTS

The Fig.2 shows gradual development of electron population throughout the device in time and space. The time dependence develops very quickly and saturates a maximum value at the carrier injection point. In space, since the electrons have same boundary properties at the both ends, the population is symmetrical in both sides of injection point. As can be seen from Eq.27, coefficients of series determine the electron population which are depend on the device active region, recombination time constant and diffusion length. Device designers should consider these coefficients for particular applications.

The time dependent part is quickly dieing out in case of $(L_n\beta_m)^2 >> 1$,

$$\text{where, } L_n\beta_m = L_n\left(\frac{\pi}{L}\right)m = \pi\left(\frac{L_n}{L}\right)m \;, m=1,2,3,... \tag{29}$$

If one is considering a fast device, he should satisfy this condition pretty well. In other words, the device should have very short diffusion length as well as very short active device area.

The time independent coefficients of series have the following form,

$$M\left(\frac{2}{L}\right)\left[\frac{\tau_n}{(L_n\beta_m)^2+1}\right] \tag{30}$$

In case of $(L_n\beta_m)^2 << 1$, one gets the following approximation,

$$\sim M\left(\frac{2}{L}\right)\times\tau_n \tag{31}$$

In case of $(L_n\beta_m)^2 >> 1$, one gets another approximation,

$$\sim M\left(\frac{2L}{\pi^2 D_n}\right)\left[\frac{1}{m^2}\right] \tag{32}$$

For present device, $L_n\beta_m = 19.2\times m \quad >> 1$, and coefficient is $\sim 2.7\,10^6\left(\frac{1}{m^2}\right)$.

For $L_n > 3L$ this approximation can be easily applied. The sum, approximately, is just a few first terms of series. On the contrary, if this condition is not satisfied, the sum will increase until the condition is satisfied. Eventually, the sum will saturate after a few terms. The convergence of series is strongly boundary value dependent. Generally, the series shows good behaviour.

IV. CONCLUSION

The exact solution of the continuity equation of minority carriers has been obtained by Green`s function approach. The solution is a convergent series of sinusoidal functions, which are well-tracked. The impact of physical parameters on device performance is clear. One can easily interpret outcome of the solution. Device designer can extract design data from the coefficient of series for particular application. The solution for any kind of generation function, nonhomogeneous boundaries, and initial value is straightforward.

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