ON ISOMORPHIC DECOMPOSITION AND CONTROLLABILITY OF SYMMETRY NONLINEAR CONTROL SYSTEMS

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ABSTRACT

This paper deals with problems on isomorphic decomposition and controllability of a kind of nonlinear systems possessing symmetries on basis of quotient systems. The isomorphic decomposition formations of these systems are drawn. Finally, it is shown that controllability of the original systems can be determined by that of subsystems, which are obtained through isomorphic decomposition. Corresponding sufficient and necessary conditions in terms of two novel theorems are derived.

I. INTRODUCTION

The differential geometric approach [1]-[4] to the study of general nonlinear and other complex systems has enabled the discovery of entirely new insight into the theory of systems and control [5]. Since the seminal paper by Isidori and his co-authors [6], early works [7]-[13] have paved the way. Following these discoveries, during the recent years, a remarkable progress has been made in the study of systems possessing symmetries in structure [14]-[17] as well as of interconnected and complex systems [18]-[20]. In general, studies of the features and the properties in nonlinear systems are much more difficult, than those in linear systems because of the complexity of the structure of nonlinear systems, of course [15], [19]. Hence, researchers have focused their attention on systems having somewhat special structure, and certain systems possessing symmetries and similarities just represent such classes of systems. In fact, the symmetric structure is rather general it can bring about a great convenience for theoretical studies and enhance their application. Therefore this approach using differential-geometric techniques has been evaluated favourably by a many scholars beginning with the linear case first [9]-[13].

The concept of symmetry for nonlinear control systems was first presented by Grizzle and Marcus [12] in 1985. They have dealt with some problems on symmetric systems such as their local and global decompositions. Study in this area has been more expanded since then. The controllability of systems possessing symmetries has hardly been studied in the past several years. Zhao and Zhang first presented the concept of general symmetry and discussed the problem of controllability in [14]. However, they did included into their study the controllability by system decomposition. The information about subsystems was not therefore used to its full.

On the grounds of the results in [14] and the general study [15] by Zhang, firstly, a concept named solvable general symmetric systems is presented in this paper and further exploited. Then the relationship between isomorphic decomposition and controllability is dealt with. On the one hand, it is considered from the angle of view on quotient systems, and, on the other hand, it is discussed from the viewpoint of feedback quotient systems. Corresponding sufficient and necessary conditions between original and decomposed systems are derived from these two points of view.

II. PROBLEM STATEMENT AND RELEVANT MATHEMATICAL DEFINITIONS

Consider the following smooth, general, nonlinear control system

$$\dot{x} = f(x,u), \tag{1}$$

where all $x \in M$, $u \in U$, and M is a smooth manifold with n-dimensions and U is a manifold admissible controls. Note that in this paper, smooth will mean the class of functions C^{∞}.

A left action (or G-action) of a connected Lie group G (with k-dimensions) on M is a smooth mapping Φ: $G \times M \to M$ such that :

for all $x \in M$, $\Phi(e,x) = x$;

for each $g, h \in G$, $\Phi(g, \Phi(h,x)) = \Phi(gh, x)$ for all $x \in M$.

The other left-action is a smooth mapping θ: $G \times M \times U \to M \times U$, $(g,x,u) \mapsto \theta_g(x,u)$. Φ is free and proper. So M/G and $Gx = \{\Phi_g x\text{: } g \in G\}$ are $n-k$-dimensional and k-dimensional manifolds respectively. Suppose further that p: $M \to M/G$ admits a cross section σ. $R_{(1)}(x)$ denotes a reachable collective (of system (1)) at point x, and $d_0 \in R_{(1)}(\sigma)$ means that each $x (\in \sigma)$ can reach d_0.

Below the necessary the definitions and concepts for dealing with system decomposition problem, some compiled from the literature and some novel ones are introduced, are presented. Certain objects such as manifolds, bundles, and distributions are not defined in the paper as they are now standard in the nonlinear systems literature [5]-[8], [12], [14]-[19]; standard mathematical references are [1]-[4].

Definition 2.1[12]: Let θ and Φ be actions of G on $M \times U$ and M respectively. Then the system (1) has symmetry (G, θ, Φ) if it commutes for all $g \in G$, where $T\Phi_g$ is the tangent map of Φ_g, π is a smooth fiber bundle and π_M is the natural projection of TM on M (see Figure 1).

Definition 2.2 [12]: (G, Φ) is a state-space symmetry of system (1) if (G, θ, Φ) is a symmetry of system (1) for $\theta_g = (\Phi_g, IDU)$: $(x,u) \mapsto (\Phi_g(x), u)$.

Definition 2.3: (G, Φ, q) is a general symmetry of system (2.1) if there exists a smooth mapping q: $G \times U \to U$, $(g,u) \mapsto q(u,g)$ such that

$$(\Phi_g)_* f(x,u) = f(\Phi_g x, q(u,g)). \tag{2}$$

Definition 2.4: A smooth mapping q: $G \times U \to U$, $(g,u) \mapsto q(u,g)$ is solvable if there exists $u' = q^*(u,g) \in U$ such that

$$q(u',g) = u, \ \forall u \in U, g \in G. \tag{3}$$

Definition 2.5: System (1) is solvable general symmetry if (G, Φ, q) is general symmetry of the system (1) and q is solvable.

Definition 2.6: The quotient system of the system (1) is the system

$$\dot{y} = \tilde{f}(y,u) = p_* f(\sigma(y), u) \tag{4}$$

defined on manifold M/G for $u \in U$.

Definition 2.7: The feedback quotient system of the system (1) is the system

$$\dot{y} = \tilde{f}'(y,u) = p_* f(\sigma(y), \alpha(\sigma(y), v)) \tag{5}$$

which is defined on manifold M/G for all $v \in V$, $\sigma(y) \in M$, where $u = \alpha(\sigma(y), v)$ is feedback law and V is a permit control manifold.

III. ISOMORPHIC DECOMPOSITION OF SYSTEMS POSSESSING SOLVABLE GENERAL SYMMETRIES

In this section, the concrete formations of isomorphic decomposition of systems possessing solvable are given. We begin with a lemma.

Lemma 3.1 [14]: Suppose the system (1) is a general symmetry system. Then

(a) $p(x(t))$ is a integral curve (starting at point $p(x_0)$) of system (4) if $x(t)$ is a integral curve (starting at point x_0) of system (1).

(b) there exists a integral curve $x(t)$ (starting at point x_0) of system (1) such that $y(t) = p(x(t))$ if $y(t)$ is a integral curve (starting at point $p(x_0)$) of system (4).

Using this lemma, it is possible to derive and prove the following conclusion.

Theorem 3.1: Suppose system (1) is a control system with general symmetry (G, Φ, q), Φ is free and proper, q is solvable, and p: $M \to M/G$ admits a cross section σ. Then system (1) is isomorphic to the system

$$\dot{y}(t) = \tilde{f}(y(t), u'(t)) = p_* f(\sigma(y(t)), u'(t)) \quad (6a)$$

$$\dot{g}(t) = \left(T_e L_{g(t)}\right)\left(T_e \tilde{\Phi}_{\sigma(y)}\right)^{-1} (f(\sigma(y(t)), u©(t)) - (T_{y(t)}\sigma) f(\sigma(y(t)), u©(t))) \quad (6b)$$

where $u' = q^*(u, g)$.

PROOF: The proof is given in the accompanied supplement due to paper size limitations.

Theorem 3.1 shows that systems possessing solvable general symmetries, under certain conditions, can be isomorphic to two subsystems. So, it is natural one to pose the questing what a relationship there may exist between the former and later system structures. It is this idea precisely that has led us to derive the result presented in the subsequent section.

IV. CONTROLLABILITY OF SYSTEMS POSSESSING SOLVABLE GENERAL SYMMETRIES

This section deals with controllability of this kind of systems with the symmetry property as mentioned above. We first introduce the following lemma.

Lemma 4.1 [14]: $\Phi_g s_t(x,u)$ is a integral curve (starting at $\Phi_g x$) of system (1) corresponding to control $q(u,g)$, that is, $\Phi_g s_t(x,u) = s_t(\Phi_g x, q(u,g))$ if $s_t(x,u)$ is a integral curve (starting at x) of the system (1) corresponding to control $u(t)$.

From all the previous presentation, Theorem 4.1 given below follows.

Theorem 4.1: Suppose that the system (1) has general symmetry (G, Φ, q), Φ is free and proper, q is solvable, and p: $M \to M/G$ admits a cross section σ. Hence x_0 $(\in M)$ is changed into g_0, y_0 through isomorphic decomposition. Further, suppose that the system (1) is weakly controllable on σ. Then by means of set $g\sigma = \{\Phi(g,x) : x \in \sigma\}, g \in G$, one can obtain that sufficient and necessary conditions of the system (1) being globally controllable at point x_0 to be determined by:

(a) subsystem (6a) is globally controllable at point y_0;

(b) subsystem (6b) is globally controllable at point g_0.

PROOF: The proof is given in the accompanied supplement due to paper size limitations.

On the grounds of Theorem 4.1, we may decompose the original system into two or more subsystems, the respective dimensions of which are decreased accordingly under certain conditions.

V. CONCLUSIONS

In this paper, the original system has been decomposed through quotient system and feedback quotient system. Each of two ways has its own advantages and disadvantages. For example, when it is very difficult for one to find an appropriate feedback law, one may decompose the original system through quotient system such as solvable general symmetry systems. It has been shown that general nonlinear control systems possessing general symmetries, under a few technical conditions, do admit isomorphic decompositions in terms of lower dimensional subsystems and feedback loops. Furthermore, controllability between the original system and the subsystems is equivalent under certain conditions. Therefore, this can be exploited when one wants to do the analysis of controllability for this kind of sym-metric systems. Their special structure can provide for a considerable convenience.

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SUPPLEMENT: *Proofs of Theorems 3.1 and 4.1*

In the first place, note the commutative diagram in Figure 1 for the considered nonlinear control system (1).

$$
\begin{tikzcd}
M \times U \arrow[rrr, "\theta_g"] \arrow[dr, "f"] \arrow[dd, "\pi"'] & & & M \times U \arrow[dl, "f"'] \arrow[dd, "\pi"] \\
& TM \arrow[r, "T\Phi_g"] \arrow[dl, "\pi_M"] & TM \arrow[dr, "\pi_M"'] & \\
M \arrow[rrr, "\Phi_g"] & & & M
\end{tikzcd}
$$

Fig.1. Commutative diagram of the system mappings

Proof of Theorem 3.1

$\sigma(y)$ is uniquely determined by y because Φ is free and proper and $p: M \to M/G$ admits a cross section σ. So, $p_* f(\sigma(y(t)), u'(t))$ is unique tangent vector of system (6a) at point $y\ (\in M/G)$ and system (6a) is uniquely decided.

Let $x_0 \in M$, $u(t)$ be a continuous time function, $x(t)$ the integral curve of system (1) corresponding to $u(t)$. And then $y(t) = p(x(t))$ is the corresponding integral curve of system (6a) having $y(0) = p(x(0))$. ***[A]***
According to $p: M \to M/G$ admitting a cross section σ, define a differentiable curve $d(t) \in M$ by $d(t) = \sigma(y(t))$. Since $p(d(t)) = p(x(t))$ and Φ is free and proper, one can write $x(t) = \Phi_{g(t)}(d(t))$ for a uniquely defined differentiable curve $g(t) \in G$.

Proof of the above statement ***[A]:*** Since system (2.1) has general symmetry (G, Φ, q), q is solvable, and $x(t)$ is the integral curve of system (1) corresponding to $u(t)$, noting that $x(t)$ and $\sigma(p(x))$ have the same orbit, one can get

$$
\begin{aligned}
\dot{y}(t) &= p_* f(x(t), u(t)) \\
&= p_* f(\Phi_{g(t)}\sigma(p(x(t))), u(t)) \\
&= p_* f(\Phi_{g(t)}\sigma(p(x(t))), q(u'(t), g(t)))
\end{aligned}
$$

that is,

$$
\begin{aligned}
\dot{y}(t) &= p_*(\Phi_{g(t)})_* f(\sigma(p(x(t))), u'(t)) \\
&= (p \circ \Phi_{g(t)})_* f(\sigma(p(x(t))), u'(t)) \\
&= p_* f(\sigma(p(x(t))), u'(t)) \\
&= p_* f(\sigma(y(t)), u'(t))
\end{aligned}
$$

where $u'(t) = q^*(u(t), g(t))$.

The goal now is to find a differential equation for $g(t)$.

From the chain rule of differentiation, one can have that

$$f(x(t), u(t)) = \dot{x} = \frac{d}{dt}\Phi(g(t), d(t)) = T_{d(t)}\Phi_{\mathrm{g(t)}}\dot{d}(t) + T_{g(t)}\Phi_{\mathrm{d(t)}}\dot{g}(t). \qquad (7)$$

The next step is to rewrite the second term. Note that $\dot{g}(t) \in T_{g(t)}G$. Let $\xi_g \in T_g G$ and denote by $\xi = T_g L_{g^{-1}}(\xi_g) \in T_e G$, where L_h is the left translation operator on G. Then for a point $m \in M$,

$$T_g\Phi_m(\xi_g) = (T_g\Phi_m)(T_e L_g)(\xi) = T_e(\Phi_m \circ L_g)(\xi) = T_e(\Phi_g \circ \Phi_m)(\xi) = (T_m\Phi_g)(T_e\Phi_m)(\xi) \quad (8)$$

But,

$$T_e\Phi_m(\xi) = \frac{d}{dt}\Phi_m(\exp t\xi)\Big|_{t=0} = \xi_M(m) \qquad (9)$$

the infinitesimal generator of Φ corresponding to ξ. Hence,

$$T_g\Phi_m(\xi_g) = T_m\Phi_g(\xi_M(m)) = T_m\Phi_g((T_g L_{g^{-1}}\xi_g)_M(m)) \qquad (10)$$

substituting (10) into (7) gives

$$f(x(t), u(t)) = T_{d(t)}\Phi_{g(t)}\dot{d}(t) + T_{d(t)}\Phi_{g(t)}(T_{g(t)}L_{g^{-1}(t)}\dot{g}(t))_M(d(x)) \qquad (11)$$

and q being solvable gives

$$f(x(t), u(t)) = f(x(t), q(u'(t), g(t))) \qquad (12)$$

which satisfies

$$u'(t) = q^*(u(t), g(t)). \qquad (13)$$

Thus, using (12) in (11) results in

$$f(x(t), q(u'(t), g(t))) = T_{d(t)}\Phi_{g(t)}\dot{d}(t) + T_{d(t)}\Phi_{g(t)}(T_{g(t)}L_{g^{-1}(t)}\dot{g}(t))_M(d(t)) \qquad (14)$$

System (1) having general symmetry (G, Φ, q) gives

$$T_m\Phi_g f(m, u'(t)) = f(\Phi_g(m), q(u'(t), g(t))), \qquad (15)$$

substituting (15) into (14). Meanwhile changing m into $d(t)$ gives

$$T_{d(t)}\Phi_{g(t)} f(d(t), u(t)) = T_{d(t)}\Phi_{g(t)}\dot{d}(t) + T_{d(t)}\Phi_{g(t)}(T_{g(t)}L_{g^{-1}(t)}\dot{g}(t))_M(d(t)) \qquad (16)$$

since $\Phi_g : M \to M$ is a diffeomorphism for all $g \in G$ and $T_{d(t)}\Phi_{g(t)}$ is nonsingular. Hence,

$$f(d(t),u'(t)) = \dot{d}(t) + (T_{g(t)} L_{g^{-1}(t)} \dot{g}(t))_M (d(t)). \tag{17}$$

Let be set

$$\xi_M (d(t)) = (T_{g(t)} L_{g^{-1}(t)} \dot{g}(t))_M (d(t)). \tag{18}$$

From (17),

$$\xi_M (d(t)) = f(d(t),u'(t)) - \dot{d}(t). \tag{19}$$

Applying (9) gives

$$T_e \Phi_{d(t)} (\xi(t)) = \xi_M (d(t)) = f(d(t),u'(t)) - \dot{d}(t). \tag{20}$$

Φ being free and proper implies that $\Phi_m\colon G \to M$ is a diffeomorphism onto its range. Hence, (20) can be solved uniquely for $\xi(t)$ to give

$$\xi(t) = (T_e \tilde{\Phi}_{d(t)})^{-1} \xi_M (d(t)), \tag{21}$$

or

$$T_{g(t)} L_{g^{-1}(t)} \dot{g}(t) = (T_e \tilde{\Phi}_{d(t)})^{-1} \xi_M (d(t)), \tag{22}$$

where $\tilde{\Phi}_m\colon G \to G \cdot m$ by $g \mapsto \Phi(g,m)$. Hence, since L_g is a diffeomorphism for all g,

$$\dot{g}(t) = (T_e L_{g(t)})(T_e \tilde{\Phi}_{\sigma(\mathrm{y(t)})})^{-1} [f(\sigma(\mathrm{y}(t)),u'(t)) - (T_{y(t)}\sigma) \tilde{f}(y(t),u'(t))]. \tag{23}$$

Finally, using the fact that $d(t) = \sigma(y(t))$, one gets

$$\dot{g}(t) = (T_e L_{g(t)})(T_e \tilde{\Phi}_{\sigma(\mathrm{y(t)})})^{-1} [f(\sigma(\mathrm{y}(t)),u'(t)) - (T_{y(t)}\sigma) \tilde{f}(y(t),u'(t))]. \tag{24}$$

substituting (13) and (6a) into (24) gives (6b).

To this end, Theorem 3.1 has been proved completely.

Proof of Theorem 4.1

Since system (1) has general symmetry, q is solvable, and $p\colon M \to M/G$ admits a cross section σ , it is obvious that system (1) is isomorphic to system (I) from Theorem 3.1 .

The necessity of Theorem 4.2: Necessity is apparent and seen quite clearly from its statement and form the above.

Before the proof of sufficiency will be completed, we prove that system (1) is weakly controllable on $g\sigma$, $\forall g \in G$, that is, $x_1 \in R_{(1)}(g\sigma)$ and $x_2 \in R_{(1)}(x_1)$ ($\forall x_1, x_2 \in g\sigma$). ***[B]***

Proof of the above statement ***[B]:*** For $\forall x_2 \in g\sigma$, there exists $x'' \in \sigma$ such that $x_2 = \Phi_g x''$. Set $x_1 = \Phi_g(x')$ for $x' \in \sigma$. Since $x' \in R_{(1)}(\sigma)$, Hence, $x' \in R_{(1)}(x'')$. From Lemma 4.1, one can easily get $x_1 = \Phi_g(x') \in R_{(1)}\Phi_g(x'') = R_{(1)}(x_2)$. For $x_2 \in g\sigma$ is at will, we have $x_1 \in R_{(1)}(g\sigma)$. The reasons of $x_2 \in R_{(1)}(x_1)$ ($\forall x_1, x_2 \in g\sigma$) is similar. Hence, system (2.1) is weakly controllable on $g\sigma$.

The sufficiency of Theorem 4.2: Let set $z \in M$ and $z \notin g_0\sigma$ (if $z \in g_0\sigma$, one can easily have $z \in R_{(1)}(x_0)$ from the conclusion having been proved just now). According to (a), there exists a integral curve $y(t)$ (starting at y_0) of system (6a) such that $y(t_1) = p(z)$. And then from Lemma 3.1, there exists a integral curve $x(t)$ (starting at x_0) of the system (1) such that $p(x(t_1)) = y(t_1) = p(z)$. It is not difficult for one to find that $x(t_1)$ and z lie on the same orbit. Therefore, there exists $\bar{g} \in G$ satisfying $z = \Phi_{\bar{g}}(x(t_1))$. From Lemma 2.4.1, one can get $z = \Phi_{\bar{g}}(x(t_1)) \in R_{(1)}(\Phi_{\bar{g}}(x_0))$.

Let $g = \bar{g}g_0$. From (b), there at least exists a point x^* $(\in g\sigma)$ $\in R_{(1)}(x_0)$ (note, otherwise, for $\forall x \in g_0\sigma$ can not reach x_1 ($\forall x_1 \in g\sigma$) since the system (1) is weakly controllable on $g_0\sigma$ and $g\sigma$). And, then one draws a contradictory conclusion that system (6b) can not satisfy g being reachable for g_0). x^* can obviously be expressed as $x^* = \Phi_g(x') = \Phi_{\bar{g}g_0}(x') = \Phi_{\bar{g}}(\Phi_{g_0}(x'))$ $(x' \in \sigma)$. Since system (1) is weakly controllable for x_0 on $g_0\sigma$, we easily know $x_0 \in R_{(1)}(\Phi_{g_0}(x'))$. And then we have $\Phi_{\bar{g}}(x_0) \in R_{(1)}(\Phi_{\bar{g}}(\Phi_{g_0}(x'))) = R_{(1)}(x^*)$ from Lemma 4.1. Hence, $z = \Phi_{\bar{g}}(x(t_1)) \in R_{(1)}(x_0)$. Finally, we obtain that the system (1) is globally controllable at point x_0 according to z $(\in M)$ being at will.