EIGENVALUE ASSIGNMENT FOR A SINGULARLY PERTURBED MINIMUM PHASE SYSTEM

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ABSTRACT

In this paper, the eigenvalue assignment of a SISO singularly perturbed problem is considered for a minimum phase system. We propose an approach for the eigenvalue assignment of a singularly perturbed minimum phase system using state feedback which leads to a simple eigenvalue reassignment procedure. The singularly perturbed system, using singular perturbation, can be divided into two subsystems independent of each other. Based on the slow subsystem *only*, we can obtain a feedback gain that reassigns the eigenvalues to provide a desired system response.

I. INTRODUCTION

Nature offers many situations of systems where more than one event occurs at different time scales. For example, an electrically driven robot manipulator can have slower mechanical dynamics and faster electrical dynamics. In such cases, we can divide the systems into two subsystems one corresponding to faster dynamics and the other corresponding to slower dynamics. Then, controllers for each one of them can be designed separately. It is then common practice to consider those events occurring at the faster scale as being instantaneous with respect to the slower ones. These results in a lesser number of variables or parameters needed to describe the evolution of the system. Several techniques have been developed in relation with such events. That is, reduction and estimation of the discrepancy between the complete system and the systems arising from the reduction. The best known methods are the averaging methods, the singular perturbation methods, and the aggregation methods [1]. Singular perturbation method has been widely used in engineering and technology problems.

Singular perturbation is a mathematical operation which can be used on class of linear/nonlinear problems where two dynamics operating on different time scales is present. In the singular perturbation method both slow and fast modes are retained, but analysis and design problems are solved in two stages [2]. Applications of singular perturbation method are found in physics, chemistry, mechanics, industrial process, and engineering [3, 4]. For example, Arino *et al.* [1] studied a model of age-structured population with two time scales. The first one is slow and corresponds to the demographic process. The second one is fast and describes the migration process between different spatial patches.

In control systems, it is always desired to enhance the performance of a given system. Knowing the relation between the closed-loop poles and the system performance, the system can be designed effectively by specifying the locations of these poles [5]. The problem of eigenvalue assignment which arises from singular system control has been studied in literature extensively [6-8].

In this paper, we consider the problem of eigenvalue assignment of a single input/ single output (SISO) singularly perturbed method for a minimum phase system. State feedback approach is considered to obtain a simple suboptimal control eigenvalue reassignment procedure. The suboptimal control is based on the slow dynamics of the system.

The organization of the paper is as follows. Section 2 presents the general model and the problem formulation. In section 3, simulation examples are considered. Section 4 presents our concluding remarks.

II. PROBLEM FORMULATION

The relationship between the input and output of an N^{th} order linear time-invariant (LTI) system is described by the differential equation

$$\sum_{i=0}^{N} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{M} b_i \frac{d^i}{dt^i} u(t) \qquad (1)$$

The system transfer function of Eq. (1) is

$$H(s)=\frac{b_M s^M + b_{M-1}s^{M-1}+\cdots+b_0}{s^N + a_{N-1}s^{N-1}+\cdots+a_0} = \frac{\sum_{i=0}^{M} b_i s^i}{\sum_{i=0}^{N} a_i s^i} \quad (2)$$

Rewriting *H(s)* as a product of terms,

$$H(s)=\frac{(b_M/a_N)\prod_{i=1}^{M}(s-c_i)}{\prod_{i=1}^{N}(s-d_i)} \quad (3)$$

then, the roots c_i are known as the zeros of *H(s)*, the roots d_i are known as the poles of *H(s),* and (b_M/a_N) is a constant gain factor. The gain factor (b_M/a_N) with the pole and zero locations in *s*-plane completely specify *H(s).*

A system is stable and causal if all of its poles are in the left half of the *s*-plane. If a system whose transfer function *H(s)* has all of its poles and zeros located in the left half of the *s*-plane, then it is said to be minimum phase [9]. The system in Eq. (1) may represent a number of physical systems which contain slow and fast modes. This system may be represented in the form of a state space model as follows.

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0)=x_0 \\ \text{y} &= Cx \end{aligned} \quad (4)$$

where x is an n-dimensional vector, u is the input signal, y is the output vector, and A and B are constant matrices with appropriate dimensions. For properly assigning the eigenvalues as desired, the following state feedback control law may be used

$$u = kx \quad (5)$$

where k is the feedback gain. Substituting Eq. (5) into Eq. (4), we obtain

$$\begin{aligned} \dot{x} &= Ax + B(kx) \\ &= (A+Bk)x \end{aligned} \quad (6)$$

Let $A_c = A + Bk$, then

$$\dot{x} = A_c x \quad (7)$$

We assume that (A, B) is controllable pair.

Now, we consider a singular perturbed linear time-invariant system [10]

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \ , \quad x_1(0)=x_{10} \\ \varepsilon \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2 u \ , \quad x_2(0)=x_{20} \\ \text{y} &= C_1 x_1 + C_2 x_2 \end{aligned} \quad (8)$$

where ε is a small positive scalar, the state x in Eq.(7) is formed by the vectors x_1 and x_2. Preliminary to a separation of slow and fast designs, system in Eq. (8) is approximately decomposed into a slow subsystem with n_1 small eigenvalues and a fast subsystem with n_2 large eigenvalues. Neglecting the fast modes is equivalent to assuming that they are infinitely fast; i.e., letting ε in Eq. (8) approaches zero. Doing so, leads to reducing the system to:

$$\begin{aligned} \dot{\bar{x}}_1 &= A_{11}\bar{x}_1 + A_{12}\bar{x}_2 + B_1\bar{u} \ , \quad \bar{x}_1(0)=x_{10} \\ 0 &= A_{21}\bar{x}_1 + A_{22}\bar{x}_2 + B_2\bar{u} \\ \bar{\text{y}} &= C_1\bar{x}_1 + C_2\bar{x}_2 \end{aligned} \quad (9)$$

where the bar indicates that $\varepsilon = 0$ for the system in Eq. (8).

Referring to Eq. (9), we see that

$$A_{22}\bar{x}_2 = -(A_{21}\bar{x}_1 + B_2\bar{u}) \quad (10)$$

Assuming that A_{22} is a full rank matrix (nonsingular), then

$$\bar{x}_2 = -(A_{22})^{-1}(A_{21}\bar{x}_1 + B_2\bar{u}) \quad (11)$$

Substituting equation (11) into equation (9) yields

$$\begin{aligned} \dot{\bar{x}}_1 &= (A_{11} - A_{22}{}^{-1}A_{21})\bar{x}_1 + (B_1 - A_{12}A_{22}{}^{-1}B_2)\bar{u} \\ \bar{\text{y}} &= (C_1 - C_2A_{22}{}^{-1}A_{21})\bar{x}_1 + (-C_2A_{22}{}^{-1}B_2)\bar{u} \end{aligned} \quad (12)$$

which can be written as

$$\begin{aligned} \dot{\bar{x}}_1 &= A_0\bar{x}_1 + B_0\bar{u} \\ \bar{\text{y}} &= C_0\bar{x}_1 + D_0\bar{u} \end{aligned} \quad (13)$$

where

$$\begin{aligned} A_0 &= A_{11} - A_{22}{}^{-1}A_{21}, \quad B_0 = B_1 - A_{12}A_{22}{}^{-1}B_2 \\ C_0 &= C_1 - C_2A_{22}{}^{-1}A_{21}, \ D_0 = -C_2A_{22}{}^{-1}B_2 \end{aligned} \quad (14)$$

Assuming that the closed loop system in Eq. (7) is controllable, the eigenvalues of the matrix A_c can be selected arbitrarily using the feedback gain k. That is, the n roots of the characteristic equation corresponding to the matrix A_c in Eq. (7)

$$det(\lambda I - A_c) = 0 \quad (15)$$

can be arbitrarily assigned where λ is an eigenvalue of the matrix A_c. Notice that *det* denotes the determinant and I denotes the identity matrix. To do that, let the feedback gain k be given as

$$k = [k_1 \ k_2 \ \cdots \ k_n] \quad (16)$$

Then the values of k_i where $i = 1,2,\ldots n$ can be obtained by comparing the characteristic polynomial in Eq. (15) with the characteristic polynomial of the desired eigenvalues model.

III. SIMULATION RESULTS

Several examples have been studied to confirm the theoretical developments. The system considered here is a single input/ single output minimum phase system. The computations were performed using MATLAB. Some results are given here.

Example: The transfer function of the available system is given as follows.

$$H(s) = \frac{s^2 + 5s + 6}{s^4 + 41.5s^3 + 380.5s^2 + 1080s + 450} \qquad (17)$$

This system has four poles and two zeros. The poles are located at –30, –6, –5, and –0.5. The zeros are located at –2 and –3. Since the poles and the zeros are in the left half of the *s*-plane, then the system is minimum phase. The pole-zero plot is shown in Figure 1. System response to a step input is shown in Figure 2.

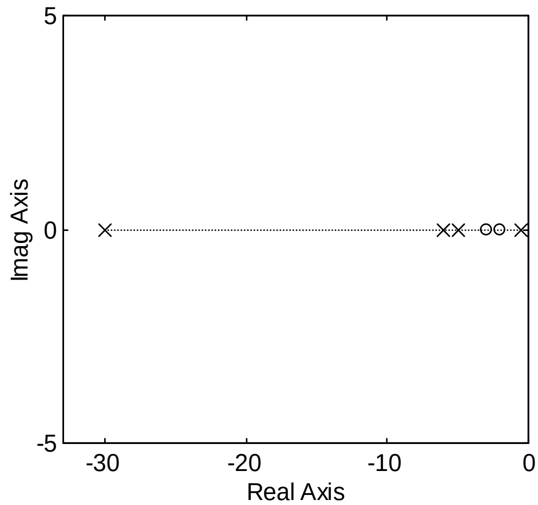

Figure 1. Pole-zero plot of the original system

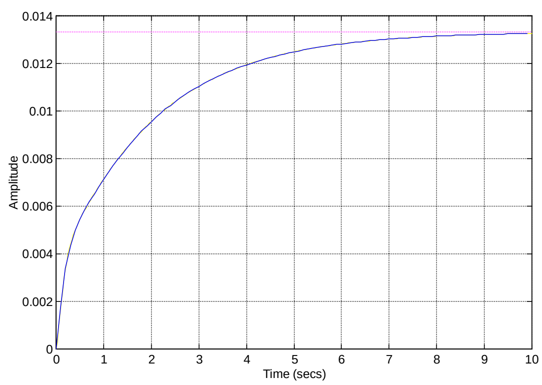

Figure 2. System response to a step function

Transforming the system in Eq. (17) to the form of a state space model, we obtain the *A, B*, and *C* matrices as follows.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -450 & -1080 & -380.5 & -41.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 6 & 5 & 1 & 0 \end{bmatrix}.$$

Applying the singular perturbation method using Equations (9) through (14) to the system in (17), we obtain the reduced order state space model.

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10.8434 & -26.0241 & -9.1687 \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 0 \\ 0.0241 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 6 & 5 & 1 \end{bmatrix}.$$

Reassigning the eigenvalues of this new model in a relatively one proper group will enhance the system performance. We have assigned the new eigenvalues to be – 2.5, –2.8, – 3.2. Notice that the important eigenvalue to be reassigned is the most dominant one which is the – 0.5. The feedback gain that places the eigenvalues as specified is obtained to be [479.6 -85.66 -27.75]. The desired eigenvalues are shown in Figure 3. Figure 4 shows the new eigenvalues after using the singular perturbation technique. It is important to mention that the new eigenvalues will not be *exactly* but close to the specified values. This is due to obtaining the feedback gain based *only* on the slow modes and ignoring the fast ones. Simulating the system with a step input after applying the reduced feedback gain to the original system shows the enhancement of the system performance, this is shown in Figure 5. The dashed line is the system before reassigning the eigenvalues and the sold line is the response after reassigning the eigenvalues.

IV. CONCLUSION

We presented an approach for eigenvalue assignment of a minimum phase system using singular perturbation. As it can be seen from the example, the new poles of the closed loop system will not be exactly but close to the assigned values. This is because the reassignment is based on the slow modes of the system only. If the eigenvalues were assigned by a closed loop feedback gain with the same system eigenvalues, the eigenvalues of the closed loop system will be a little different, but the response will be the same. Hence, even though the eigenvalues are not assigned exactly as specified they provide the response as if the system does have those assigned eigenvalues.

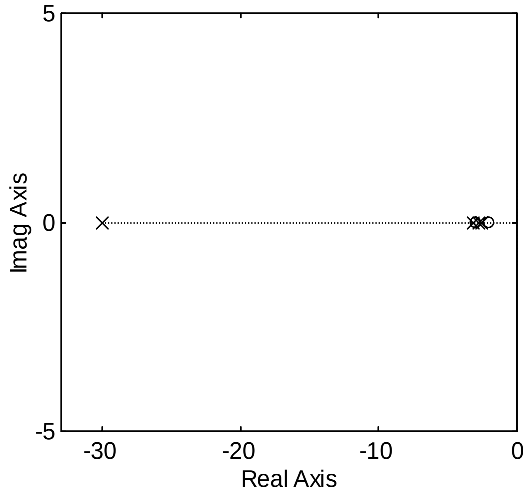

Figure 3. Desired Eigenvalues

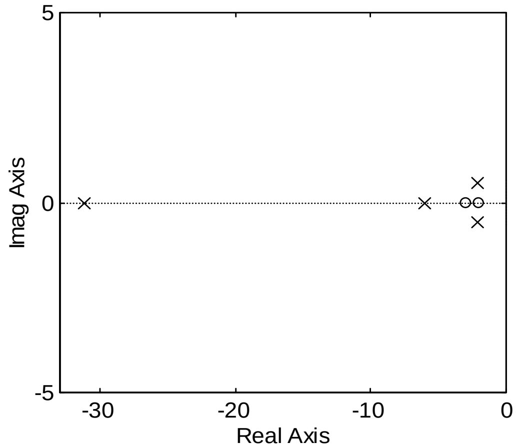

Figure 4. Obtained Eigenvalues

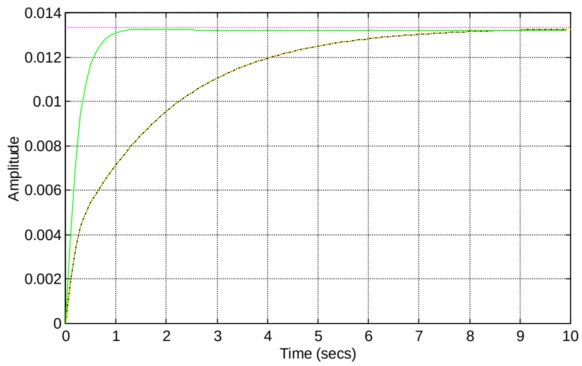

Figure 5. Normalized system response to a step function after eigenvalue assignment

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