Zero Error Split Step FDTD Method for Narrow Band Applications

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Abstract

It has recently been reported that the four split step finite difference time domain (4 SS FDTD) method is isotropic at a certain stability factor for each space step value. It is also known that the average value of the numerical phase velocity of FDTD methods can be corrected for given stability factor and space step value. In this paper polynomial expressions are obtained for the stability factor values giving zero numerical phase velocity anisotropic error and zero error average numerical phase velocity. These polynomial expressions are included in the 4 SS FDTD codes, so that once space value is chosen from simulation considerations the stability factor (or the time step) and the numerical phase velocity correction factor is obtained directly. A performance study of the method show that zero error isotropic numerical phase velocity can be obtained at space step values as large as five cells per wavelength. The method is narrowband for large space step values.

1. Introduction

Finite difference time domain (FDTD) method and its variations have proved to be very powerful in solving electromagnetic problems [1]. The FDTD method is not error free. Errors due to the discretization of the space and time domains cause the phase velocity of the waves for numerical simulations to differ from physical phase velocity (v) in the medium of concern. The magnitudes of these errors are not constant and vary with space discretization step size, time discretization step size, the frequency of the wave and direction of propagation of the wave. These errors cause dispersion and delay in the numerical waves.

There have been several publications for reducing the average level or the anisotropy of these errors [2,3]. The simplest way of reducing the average level of these errors is to multiply v of the medium (or ε and μ of the medium) with an appropriate constant to bring the average error to zero [2]. This is a very simple and effective technique but it has several drawbacks. The anisotropic error remains the same, which means that the phase velocity is still has large errors in certain directions. One other disadvantage of this type of correction is that the compensation is made only for chosen values of the wave and domain parameters (frequency, space step size and time step size). If one of these parameters is changed the multiplication constant for the c should be changed as well. As the compensation is made for a single frequency this compensation may not be not appropriate for wide band wave pulses.

A recent publication on a 4 split step unconditionally stable FDTD method by the authors of this paper [4] reports that there is a certain stability factor for each space step size such that the numerical phase velocity is isotropic. In this paper we show that when we use the 4 split step FDTD method of [4] and make a correction for the average level of the phase velocity with the right choice of parameters, we can have zero average and zero anisotropic numerical phase velocity error at a certain frequency.

2. Zero Error Numerical Phase Velocity

For simplicity the study has been carried out for 2-D TM wave propagation in a lossless, homogenous medium. The numerical dispersion relationship of the 4 split step FDTD method (4 SS FDTD) is given by [4]:

$$\cos(\omega\Delta t) = \zeta \tag{1}$$

where $\zeta = \alpha/\beta$ and α and β are given by:

$$\begin{aligned}\alpha = {} & 256 - 384(v\Delta t)^2(\eta_x^2 + \eta_y^2) \\ & + 16(v\Delta t)^4(\eta_x^4 + \eta_y^4 + 4\eta_x^2\eta_y^2) \\ & - 24(v\Delta t)^6(\eta_x^4\eta_y^2 + \eta_x^2\eta_y^4) + (v\Delta t)^8(\eta_x^4\eta_y^4)\end{aligned} \tag{2}$$

$$\begin{aligned}\beta = {} & 256 + 128(v\Delta t)^2(\eta_x^2 + \eta_y^2) \\ & + 16(v\Delta t)^4(\eta_x^4 + \eta_y^4 + 4\eta_x^2\eta_y^2) \\ & + 8(v\Delta t)^6(\eta_x^4\eta_y^2 + \eta_x^2\eta_y^4) + (v\Delta t)^8(\eta_x^4\eta_y^4)\end{aligned} \tag{3}$$

ω is the angular frequency, v is the speed of light in the medium, Δt is the time increment and

$$\eta_\gamma = -\frac{1}{\Delta\gamma}\sin\left(\frac{\tilde{k}_\gamma\Delta\gamma}{2}\right) \tag{4}$$

for $(\gamma \in x, y)$. Here $\Delta\gamma$ represents the space step size and $\tilde{k}_\gamma$ represents the numerical wavenumber for the corresponding direction. The equation (1) can be solved by using iteration methods to obtain the numerical propagation velocity in the computational domain.

It is shown in [4] that for each space step value, Δ, ($\Delta = \Delta x = \Delta y$), there exists a stability factor value, s, for which the anisotropic error is zero. The stability factor s is defined as:

$$s = \frac{v\Delta t}{\Delta} \tag{5}$$

and the anisotropic error is defined as:

$$Anisotropic\ error = \frac{max\{\tilde{v}(\phi)\} - min\{\tilde{v}(\phi)\}}{min\{\tilde{v}(\phi)\}} \tag{6}$$

The s values which give zero anisotropic error are plotted against Δ in Fig. 1. The average value of the numerical phase velocity (*npv*) for the 4 split step FDTD method, as in similar methods, occurs at $\phi = 22.5^{\circ}$. So, the average numerical phase velocity can be obtained at each Δ and the corresponding zero anisotropic error s value. The normalized average numerical phase velocity values are also shown in Fig. 2. Once the Δ value is determined from simulation requirements, we can use Fig. 1 and 2 to obtain the zero anisotropic s value (or Δt) and the *npv* correction factor (the inverse of the normalized average numerical phase velocity). The physical phase velocity is than multiplied with this correction factor. Using these s and corrected phase velocity values we have a zero error FDTD method at the design frequency.

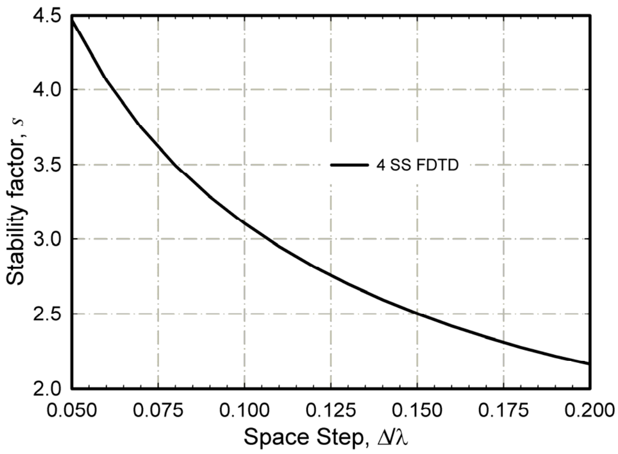

Fig. 1. Zero anisotropic error s values against the space step.

Alternatively we can obtain polynomial expressions for s and *npv* in terms of Δ. Five term expressions for these functions obtained by using Polyfit function of Matlab, for the range Δ=λ/5 to Δ=λ/20, where λ represents the wavelength, are given below:

$$\begin{aligned} s = 8.25 - 119.55\Delta + 1141.32\Delta^2 \\ - 6111.05\Delta^3 + 16873.60\Delta^4 - 18572.63\Delta^5 \end{aligned} \tag{7}$$

$$\begin{aligned} npv = 0.93 + 2.43\Delta - 60.97\Delta^2 \\ + 512.41\Delta^3 - 2104.26\Delta^4 + 3259.82\Delta^5 \end{aligned} \tag{8}$$

Equations (7) and (8) can be incorporated in the codes of the 4 SS FDTD method so that once the Δ value is chosen from simulation considerations the optimum s (or Δt) and *npv* correction factor are obtained and the corrected phase velocity is calculated directly. This way it will not be necessary to find s and *npv* correction factor values each time the grid size (Δ) is changed.

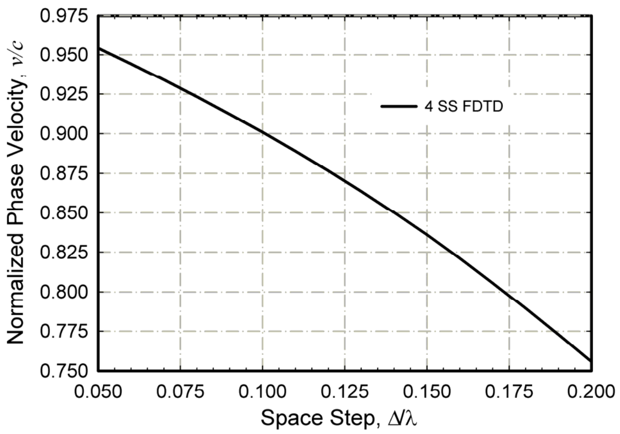

Fig. 2. Zero normalized numerical phase velocity (*npv*) against the space step, for s values correspond to Fig. 1.

3. Performance Study

The performance of the method is compared with the performances of the conventional FDTD and 2 split step unconditionally stable FDTD (2 SS FDTD) methods (ADI [5] and LOD [6, 7] methods).

Fig. 3 shows the normalized numerical phase velocity against angle for the conventional FDTD, 2 SS FDTD, uncorrected and corrected 4 SS FDTD methods at Δ=λ/5. The time step (or s) value of the 4 SS FDTD method is taken to be the value which gives zero anisotropic error for Δ=λ/5 (s=2.16).

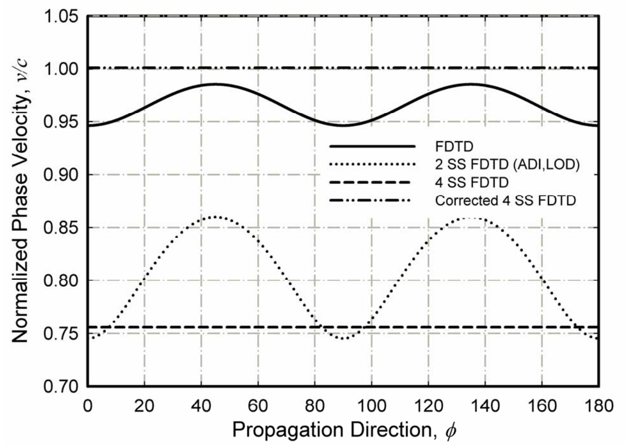

Fig. 3. Normalized numerical phase velocity (*npv*) against the propagation direction, for Δ=λ/5 and s=0.54 for FDTD, s=1.08 for 2 SS FDTD, s=2.16 for 4 SS FDTD.

The s values for the other two methods are chosen such that the split step time step size of the 2 SS FDTD and 4 SS FDTD methods are the same as the time step of the conventional FDTD method (i.e. s=0.54 for conventional FDTD, s=1.08 for 2 SS FDTD). This choice of s values ensures that the total number of time steps (including sub steps) is the same for all the methods. We can observe from Fig. 3 that the corrected 4 SS FDTD has zero error at all angles. It is obvious that if the average *npv* are corrected for conventional FDTD and 2 SS FDTD methods, they would still have large anisotropic errors having unacceptably large dispersion errors at certain angles.

Fig. 4 shows the normalized numerical phase velocity against angle for the 2 SS FDTD, corrected and uncorrected 4 SS FDTD methods at $\Delta=\lambda/20$. The s value (or Δt) of the 4 SS FDTD method is taken to be the value which gives zero anisotropic error for $\Delta=\lambda/20$ (s=4.47). The s value for the 2 SS FDTD method (s=2.24) is chosen such that the split step time step size of the 2 SS FDTD and 4 SS FDTD methods are the same. As the conventional FDTD method becomes unstable at the corresponding time step value we could not include it in this figure. We can observe from Fig. 4 that the corrected 4 SS FDTD has zero error at all angles while the 2 SS FDTD method has unacceptably large anisotropic error for this space step size as well.

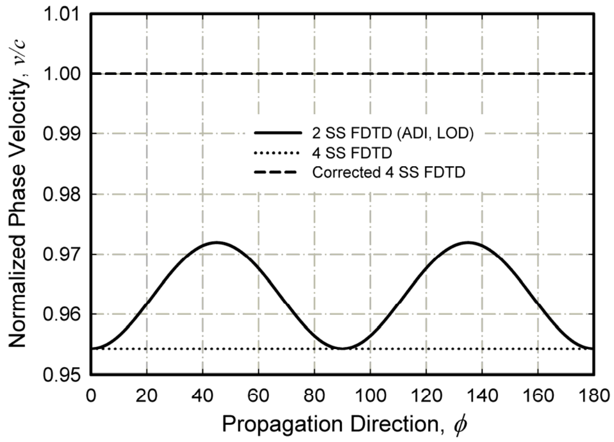

Fig. 4. Normalized numerical phase velocity (*npv*) against the propagation direction, for $\Delta=\lambda/20$ and s=2.24 for 2 SS FDTD, s=4.47 for 4 SS FDTD.

The zero error performance of the 4 SS FDTD method is only valid at the design frequency. When the frequency is different the method will have some anisotropic and average errors. Fig. 5 shows the average error against normalized frequency and Fig. 6 shows anisotropic error against normalized frequency for the above three methods. Here Δ is equal to $\lambda/5$ at the design frequency, f_0, and the s values correspond to the values in Fig. 3. We can observe that there are variations in *npv* and anisotropic errors as the frequency changes. If we take 1% error as the acceptable error, the bandwidth due to the average error (Fig. 5) would be about 4% for the 4 SS FDTD method, while the bandwidth due to the anisotropic error (Fig. 6) would be much wider. Although the conventional FDTD method would have smaller average error over much wider bandwidth if it is corrected at the design frequency, its anisotropic error would still be very large.

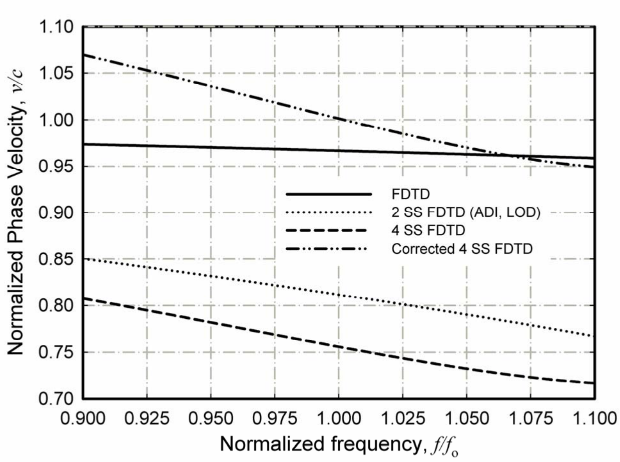

Fig. 5. Normalized numerical phase velocity (*npv*) against the normalized frequency (f/f_o), s=0.54 for FDTD, s=1.08 for 2 SS FDTD, s=2.16 for 4 SS FDTD.

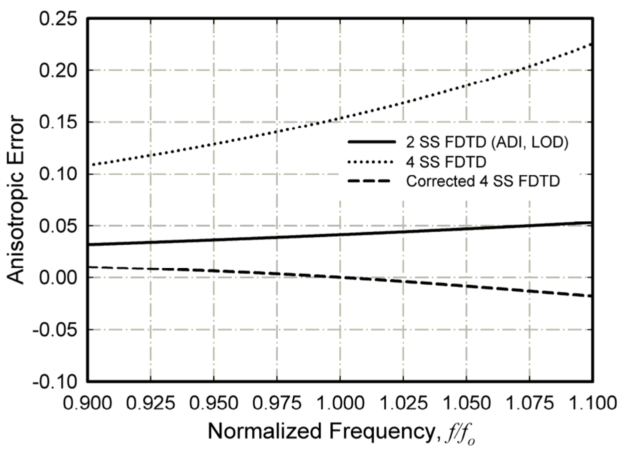

Fig. 6. Anisotropic error against the normalized frequency (f/f_o), s=0.54 for FDTD, s=1.08 for 2 SS FDTD, s=2.16 for 4 SS FDTD.

Fig. 7 shows the average error against frequency and Fig. 8 shows anisotropic error against frequency for the for the 2 SS FDTD and 4 SS FDTD methods. Here Δ is equal to $\Delta=\lambda/20$ at the design frequency, f_0, and s values correspond to the values in Fig. 4. We can observe that the variations in *npv* and anisotropic errors with frequency is smaller at this Δ. If we take 1% error as the acceptable error, the bandwidth due to the average error (Fig. 7) would be about 20% for the 4 SS FDTD method, while the bandwidth due to the anisotropic error (Fig. 6) would be much wider, while the 2 SS FDTD method has large anisotropic error. When the performance of the method is studied at Δ values larger than $\lambda/5$ the bandwidth becomes even narrower.

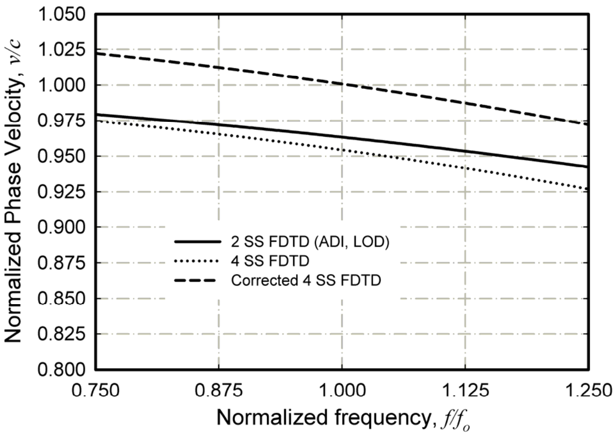

Fig. 7. Normalized numerical phase velocity (*npv*) against the normalized frequency (f/f_o), s=2.24 for 2 SS FDTD, s=4.47 for 4 SS FDTD.

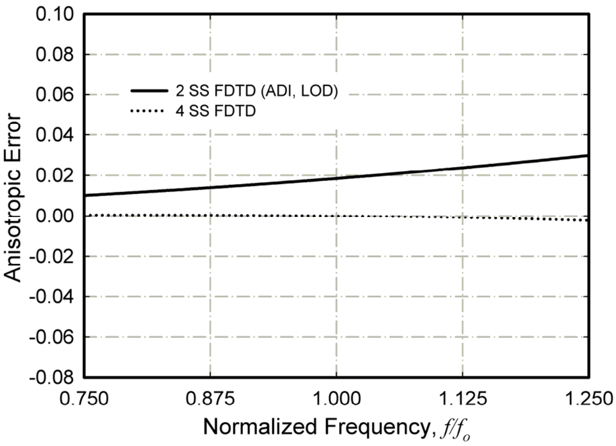

Fig. 8. Anisotropic error against the normalized frequency (f/f_o), s=2.24 for 2 SS FDTD, s=4.47 for 4 SS FDTD.

6. Conclusions

The fact that the four split step finite difference time domain method is isotropic at a certain stability factor and space step values are used with a correction factor to obtain a zero error FDTD method. Once the space step size is determined from simulation considerations the optimum stability factor and normalized phase velocity correction factor are obtained and the corrected phase velocity is calculated directly. The performance study show that isotropic error performance is possible at even very large space step values, but the method is narrow band at larger space step values.

7. References

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